Formal Logic
AKA
Symbolic Logic

We will be working problems, so please have paper and a writing instrument on hand.

Course Outline

Day 1

- Housekeeping
- Propositional Logic

Day 2

- Prepositional Logic

Day 3

- Logic Problems

Scope

Informal Logic

Set of techniques used to evaluate arguments made in everyday language.

Formal Logic

Set of formulae used to assign truth values to symbolic equations.

Scope

Informal Logic

Set of techniques used to evaluate arguments made in everyday language.

Formal Logic

Rules of inference and replacement used to assign truth values to symbolic equations.

Why Bother?

- Eliminate ambiguity
- Remove bias
- Nullify uncertainty

Transform messy problems of language into clear, elegant, and unambiguous math problems.

Abductive, Inductive, and Deductive Reasoning

Abductive

Start with a concrete instance.
Draw a conclusion based on the best explanation.

My lettuce plants nearly always die in January after a cold night.
I hypothesize frosts in January kill my plants.

Abductive, Inductive, and Deductive Reasoning

Inductive

Start with an hypothesis.
Find supporting evidence.
Find contradictory evidence.
Generalize.

Frost kills lettuce plants in January.
True in 1970 - 2002, and 2004 - 2021.
False in 2003.
For practical purposes, assume lettuce will not survive through January.

Abductive, Inductive, and Deductive Reasoning

Deductive

Start with what you know.
Draw a conclusion.

Frost usually kills lettuce in January.
Lettuce takes 60 days to produce a crop.
Therefore, for reliable results, plant lettuce no later than October.

Abductive, Inductive, and Deductive Reasoning

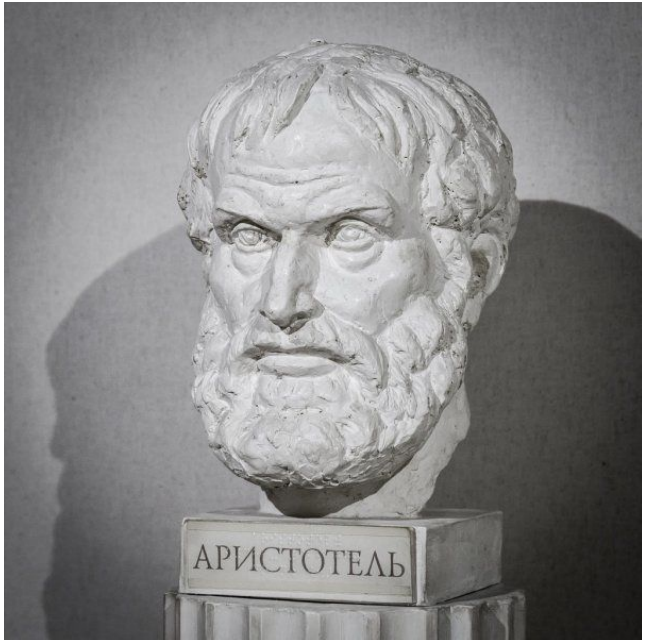
Deductive

Start with what you know.
Draw a conclusion.

Frost usually kills lettuce in January.
Lettuce takes 60 days to produce a crop.
Therefore, for reliable results, plant lettuce no later than October.

The domain of this course is **deductive logic**.

Where did it all Begin?



Aristotle, 384 - 322 B.C.E.

Socrates -> Plato -> Aristotle

- Three laws of thought.
  - Excluded Middle
  - Non-Contradiction
  - Identity
- The syllogism as the basis of deductive logic.

The Proposition

A statement to which a truth-value of true or false can be assigned.

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All humans are mortal.

All dogs are human.

What is my dog’s name?

Brutalist Architecture is ugly.

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A statement to which a truth-value of true or false can be assigned.

All humans are mortal. ✓

All dogs are human. ✓

What is my dog’s name? ✗

Brutalist Architecture is ugly. ✗

Symbols

Let p represent the proposition, “All humans are mortal.”

Let q represent the proposition, “All dogs are human.”

Use p, q, … in place of the full proposition when defining logic equations.

Truth Values

Let p be a proposition.

Then p **must** evaluate to **exactly one** element from the set {True, False}.

Excluded Middle

In math-speak, propositions are the domain and {True, False} is the range, where the function is one-to-one.

Truth Values

p
True
False

Truth Tables

Peirce (1839 - 1914) or Wittgenstein (1889 - 1951)

Truth tables list all possible values of propositions and derived propositions.

Given n propositions, there are 2^n rows in the corresponding truth table.

p
True
False

p	q
True	True
True	False
False	True
False	False

Logical Functions

Not, Negation

Let p be a proposition.

~p, !p, ¬p, Not(p), $\overline{p}$ all denote the negation function applied to p.

p	~p
True	False
False	True

Logical Functions

Or, Disjunction

Let p, q be propositions

p+q, p∨q, p or q, OR(p,q) denote disjunction applied to p, q.

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Logical Functions

p∨q is False **only** when both p and q are false.

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Linguistic versus Logical OR

“Are you going today, **or** tomorrow?”

Linguistic versus Logical OR

“Are you going today, **or** tomorrow?”

“Today.” or perhaps, “Tomorrow.” Maybe even, “Neither.” But probably not “yes” or “no.”

This is not a proposition. For the sake of your interpersonal relationships, do not answer, “yes.”

Excluded Middle

A proposition must have exactly one value from the set {True, False}.

p	~p	p ∨ ~p
True	False	True
False	True	True

Excluded Middle

A proposition must have exactly one value from the set {True, False}.

p	~p	p ∨ ~p
True	False	True
False	True	True

A **tautology** is a logic function that always evaluates to True.

Logical Functions

And, Conjunction

Let p, q be propositions

pq, p·q, $p \wedge q$, p AND q, AND(p,q) denote conjunction applied to p, q.

p	q	$p \wedge q$
True	True	True
True	False	False
False	True	False
False	False	False

Logical Functions

p $\wedge$ q is true only when both p and q are true.

p	q	p $\wedge$ q
True	True	True
True	False	False
False	True	False
False	False	False

Logical Functions

p ∧ q is true only when both p and q are true.

p	q	p ∧ q
True	True	True
True	False	False
False	True	False
False	False	False

The function p ∧ q is sometimes True and sometimes False. This is called a **contingency**.

Linguistic Versus Logical AND

"Rainy days and Mondays always get me down."

Linguistic Versus Logical AND

“Rainy days and Mondays always get me down.”

Rainy days get me down.
Mondays get me down.
Rainy Mondays get me down.

Lingustic ‘and’ often means logical OR.

Non-Contradiction

p and ~p can never both be true.

p	~p	$p \wedge \sim p$
True	False	False
False	True	False

Non-Contradiction

p and ~p can never both be true.

p	~p	$p \land \sim p$
True	False	False
False	True	False

A **contradiction** is a logic function that always evaluates to False.

Logical Functions

Implication, If/Then

Let p, q be propositions

$p \to q$, $p \Rightarrow q$, $p \supset q$, IF p THEN q, denote implication applied to p, q.

p	q	$p \to q$
True	True	True
True	False	False
False	True	True
False	False	True

Note: p is called the premise and q the conclusion.

Logical Functions

p→q is false **only** when the premise is true and the conclusion false.

p	q	p→q
True	True	True
True	False	False
False	True	True
False	False	True

Logical Functions

p→q is false only when the premise is true and the conclusion false.

p	q	p→q
True	True	True
True	False	False
False	True	True
False	False	True

“False implies anything.”

Linguistic Versus Logical IF/Then

If I buy this car then I will overdraw my checking account.

The statement is linguistically meaningful only when the premise (If I buy this car) is true.

Similar to a logical AND rather than a logical IF/THEN.

Identity

p→p or p = p.

p	p→p
True	True
False	True

p→p is a tautology.

Closure

The proposition domain is closed under ~,∨,∧,→.

That is,

If p is a proposition, then ~p is a proposition.
If p,q are propositions, then p∨q is a proposition.
If p,q are propositions, then p∧q is a proposition.
If p,q are propositions, then p→q is a proposition.

Examples

In the following examples, convert the text into symbolic logic statements, then assign a truth value to each resulting statement.

Examples

In the following examples, convert the text into symbolic logic statements, then assign a truth value to each resulting statement.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Examples

Step 1: define propositions and construct the logic statement.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Let p represent the statement, “My cat speaks French.”

Let q represent the statement, “Williamsburg is the capital of Virginia.”

Examples

Step 1: define propositions and construct the logic statement.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Let p represent the statement, “My cat speaks French.”

Let q represent the statement, “Williamsburg is the capital of Virginia.”

$\sim p \lor q$

Examples

Step 2: Evaluate the statement given the propositions' truth values.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

p = False

q = False

~(False) ∨ False = True ∨ False = True

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

Let p be, “My cat speaks French.”

Let q be, “Williamsburg is the capital of Virginia.”

$p \rightarrow q$

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

p = False

q = False

False→False = True

Order of Operations

When evaluating logic equations, perform functions in this order, from left to right.

1. Parentheses
2. Negation
3. Conjunction
4. Disjunction
5. Implication

Example

Assume a,b are True and x is False.

Evaluate the following logic formula.

$\sim(ab) \rightarrow (a \vee x)$

Example

Assume a,b are True and x,y are False.

Evaluate the following logic formula.

~(ab)→(a∨x)
~True→(a∨x)
~True→True
False→True
True

Examples

Assume a,b are True and x,y are False.

Evaluate the following logic formula.

$(a \to b) \to \sim(a \lor x \to y)$

Examples

Assume a,b are True and x,y are False.

Evaluate the following logic formula.

(a→b)→~(a∨x→y)
True→~(a∨x→y)
True→~(True→y)
True→~False
True→True
True

Examples

Assume a,b are True and x,y are False.
Evaluate the following logic formula.

$$\sim x \vee y \wedge a \rightarrow (b \rightarrow \sim(a \vee x))$$

Examples

Assume a,b are True and x,y are False.
Evaluate the following logic formula.

~x∨y∧a→(b→~(a∨x))
~x∨y∧a→(b→~True)
~x∨y∧a→(b→False)
~x∨y∧a→False
True∨y∧a→False
True∨False→False
True→False
False

Replacement or Equivalence Rules

Use these rules to simplify or rearrange propositions.

- Constant Evaluation
- Commutativity
- Associativity
- Distribution
- Double Negation
- Tautology
- Material Implication
- De Morgan's

Constant Evaluation

Let p be a proposition.

$False \wedge p = False$
$False \vee p = p$
$True \wedge p = p$
$True \vee p = True$
$p \wedge \sim p = False$ (Non-Contradiction)
$p \vee \sim p = True$ (Excluded Middle)
$p \to p = True$ (Identity)
$False \to p = True$ (False Implies Anything)

Commutativity

Let p,q be propositions.

Then $p \vee q = q \vee p$

Then $p \wedge q = q \wedge p$

p	q	$p \vee q$	$q \vee p$
True	True	True	True
True	False	True	True
False	True	True	True
False	False	False	False

p	q	$p \wedge q$	$q \wedge p$
True	True	True	True
True	False	False	False
False	True	False	False
False	False	False	False

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Commutativity

Let p,q be propositions.

Then $p \to q \neq q \to p$. (I.e. implication is not commutative.)

p	q	$p \to q$	$q \to p$
True	True	True	True
True	False	False	True
False	True	True	False
False	False	True	True

Associativity

Let p,q,r be propositions.

Then $(p \lor q) \lor r = p \lor (q \lor r)$

p	q	r	$p \lor q$	$q \lor r$	$(p \lor q) \lor r$	$p \lor (q \lor r)$
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	True	True	True
True	False	False	True	False	True	True
False	True	True	True	True	True	True
False	True	False	True	True	True	True
False	False	True	False	True	True	True
False	False	False	False	False	False	False

Associativity

Let p,q,r be propositions.

Then $(p \land q) \land r = p \land (q \land r)$

p	q	r	$p \land q$	$q \land r$	$(p \land q) \land r$	$p \land (q \land r)$
True	True	True	True	True	True	True
True	True	False	True	False	False	False
True	False	True	False	False	False	False
True	False	False	False	False	False	False
False	True	True	False	True	False	False
False	True	False	False	False	False	False
False	False	True	False	False	False	False
False	False	False	False	False	False	False

Drop the Parentheses

Let $p_1, p_2, \ldots, p_n$ be propositions.

Then $(((p_1 \vee p_2) \vee p_3)...p_n) = p_1 \vee p_2 \vee p_3 ... \vee p_n$

Then $(((p_1 \wedge p_2) \wedge p_3)...p_n) = p_1 \wedge p_2 \wedge p_3 ... \wedge p_n = p_1 p_2 p_3 ... p_n$

Drop the Parentheses

Let p_1, p_2, …, p_n be propositions.

Then $(((p_1 \vee p_2) \vee p_3)...p_n) = p_1 \vee p_2 \vee p_3 ... \vee p_n$

Then $(((p_1 \wedge p_2) \wedge p_3)...p_n) = p_1 \wedge p_2 \wedge p_3 ... \wedge p_n = p_1 p_2 p_3 ... p_n$

This only works with all $\wedge$ or all $\vee$. In a mix, retain the parentheses.

Example

Assume p,q,r are propositions where p is false, q is false, and r is true.

Evaluate $p \land (q \lor r)$ and $p \land q \lor r$.

$p \land (q \lor r)$	$p \land q \lor r$
$p \land$True	False$\lor r$
False$\land$True	False$\lor$True
False	True

Associativity

(p→q)→r ≠ p→(q→r)

p	q	r	p→q	q→r	(p→q)→r	p→(q→r)
True	True	True	True	True	True	True
True	True	False	True	False	False	False
True	False	True	False	True	True	True
True	False	False	False	True	True	True
False	True	True	True	True	True	True
False	True	False	True	False	False	True
False	False	True	True	True	True	True
False	False	False	True	True	False	True

Distribution

Let p,q,r be propositions.

Then $p(q \vee r) = pq \vee pr$

Distribution

p	q	r	q∨r	p(q∨r)	pq	pr	pq∨pr
True	True	True	True	True	True	True	True
True	True	False	True	True	True	False	True
True	False	True	True	True	False	True	True
True	False	False	False	False	False	False	False
False	True	True	True	False	False	False	False
False	True	False	True	False	False	False	False
False	False	True	True	False	False	False	False
False	False	False	False	False	False	False	False

Distribution

Let p,q,r be propositions.

Then $p \vee (qr) = (p \vee q)(p \vee r)$

Distribution

p	q	r	qr	$p \vee (qr)$	$p \vee q$	$p \vee r$	$(p \vee q)(p \vee q)$
True	True	True	True	True	True	True	True
True	True	False	False	True	True	True	True
True	False	True	False	True	True	True	True
True	False	False	False	True	True	True	True
False	True	True	True	True	True	True	True
False	True	False	False	False	True	False	False
False	False	True	False	False	False	True	False
False	False	False	False	False	False	False	False

Example

Rewrite the logic statement to remove the parentheses.

(p∨q)(r∨~q)	
x(r∨~q)	Let x = (p∨q)
rx∨~qx	Dist.
r(p∨q)∨~q(p∨q)	Substitution
rp∨rq∨~qp∨~qq	Dist.
rp∨rq∨~qp∨False	Non-Contradiction
rp∨rq∨~qp	Const. Eval.

Double Negation

Let p be a proposition.

Then ~~p = p.

p	~p	~~p
True	False	True
False	True	False

Tautology

Let p be a proposition.

Then p = p∨p and p = p∧p.

p	p∨p	p∧p
True	True	True
False	False	False

Material Implication

Let p, q be propositions.

Then $p \to q = \sim p \vee q$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
True	True	True	False	True
True	False	False	False	False
False	True	True	True	True
False	False	True	True	True

De Morgan’s Theorem

De Morgan (1806 - 1871) Professor of Mathematics at University College, London.

- Formalized Mathematical Induction.
- Investigated informal negation rules for inclusion in formal logic.

De Morgan's Theorem

Let p, q be propositions.

Then $\sim(pq) = \sim p \vee \sim q$, and

$\sim(p \vee q) = \sim p \wedge \sim q$

De Morgan’s Theorem

Let p, q be propositions.

Then ~(pq) = ~p⋁~q.

p	q	pq	~(pq)	~p	~q	~p⋁~q
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

De Morgan’s Theorem

Let p, q be propositions.

Then ~(p∨q) = ~p∧~q.

p	q	p∨q	~(p∨q)	~p	~q	~p~q
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Example

Remove the parentheses and simplify the following statement.

~(p∨r)→~(qr)	

Example

Remove the parentheses and simplify the following statement.

~(p∨r)→~(qr)	
~~(p∨r)∨~(qr)	Material Implication
(p∨r)∨~(qr)	Double Negation
(p∨r)∨(~q∨~r)	De Morgan's
p∨~q∨(r∨~r)	Comm./Assoc.
p∨~q∨True	Excluded Middle
True	Constant Evaluation
∴~(p∨r)→~(qr) is a tautology	

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using replacement rules and rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Example

Determine if the following formula is a contradiction, a tautology, or a contingent statement.

$q \rightarrow (p \vee \sim q)$	

Example

Determine if the following formula is a contradiction, a tautology, or a contingent statement.

q→(p∨~q)	
~q ∨ (p ∨ ~q)	Material Implication
~q ∨ (~q ∨ p)	Comm.
(~q ∨ ~q) ∨ p	Assoc.
~q∨p	Identity

Example

~q∨p

p	q	~q	~q∨p
True	True	False	True
True	False	True	True
False	True	False	False
False	False	True	True

∴q→(p∨~q) is a contingent.

The Syllogism

Major Premise	Relate two propositions and assert the relationship is True.
Minor Premise	Assert the truth of a value or functional relationship of a proposition from the major premise.
Conclusion	Derive the truth value or relationship of the second proposition.

Modus Ponens

If p then q p Therefore q	$p \to q$ p $\therefore q$

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

Modus Ponens

Modus Ponens is a tautology.

p	q	p→q
True	True	False
True	False	True
False	True	False
False	False	True

Modus Ponens

Modus Ponens is a tautology.

$p \rightarrow q$ is True

p	q	$p \rightarrow q$
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

Modus Ponens

Modus Ponens is a tautology.

p is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
~~False~~	~~False~~	~~True~~

Modus Ponens

Modus Ponens is a tautology.

p is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
~~False~~	~~False~~	~~True~~

Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

We’ve had 1.3 cm of rain in the past week,
Therefore I don’t need to water my garden.

Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.
We’ve had 1.3 cm of rain in the past week,
Therefore I don’t need to water my garden.

Let p be the proposition, “We’ve had at least 1 cm of rain in the past week.”
Let q be the proposition, “I need to water my garden.”

$p \to \sim q$
p
$\therefore \sim q$

Modus Tollens

If p then q Not q Therefore Not p	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

All humans are mortal.
Socrates is not mortal,
Therefore Socrates is not a human.

Modus Tollens

Modus Tollens is a tautology.

p→q is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

Modus Tollens

Modus Tollens is a tautology.

q is False (q is not True)

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
False	False	True

Modus Tollens

Modus Tollens is a tautology.

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
False	False	True

Modus Tollens Examples

If we've had at least 1 cm of rain in the past week, I don't need to water my garden.
I need to water my garden,
Therefore we've had less than 1 cm of rain in the past week.

Let p be the proposition, "We've had at least 1 cm of rain in the past week."
Let q be the proposition, "I need to water my garden."

Modus Tollens Examples

Let p be the proposition, “We’ve had at least 1 cm of rain in the past week.”
Let q be the proposition, “I need to water my garden.”

$p \rightarrow \sim q$
q
$\therefore \sim p$

Modus Tollens Common Mistakes

- Assuming ~p and drawing a conclusion.
- Assuming q and drawing a conclusion.

Modus Tollens Common Mistake

If p then q
Not p
Therefore Not q

All humans are mortal.
Socrates is not human,
Therefore Socrates is not Mortal.

Modus Tollens Common Mistake

If p then q
Not p
Therefore Not q

All humans are mortal.
Socrates is not human,
Therefore Socrates is not Mortal.

Socrates might be a cat, or an elephant, or a fish, or a mushroom, or a…
“False implies anything.”

Modus Tollens Common Mistake

Modus Tollens is a tautology.

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

"False implies anything."

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

We haven’t had 1 cm of rain in the past week,

Therefore I need to water my garden.

Modus Tollens Mistake Example

If we've had at least 1 cm of rain in the past week, I don't need to water my garden.

We haven't had 1 cm of rain in the past week,

Therefore I need to water my garden.

Modus Tollens Mistake Example

If p then q
q
∴p

All humans are mortal.
Socrates is mortal,
Therefore Socrates is a human.

Modus Tollens Mistake Example

If p then q

q

∴p

All humans are mortal.

Socrates is mortal,

Therefore Socrates is a human.

Socrates might be a turtle, a jellyfish, a fern...

Modus Tollens Common Mistake

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

I don’t need to water my garden,

Therefore we’ve had at least one cm of rain in the past week.

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

I don’t need to water my garden,

Therefore we’ve had at least one cm of rain in the past week.

Perhaps I already watered my garden yesterday. Perhaps it is winter. Perhaps...

Disjunctive Syllogism

$p \lor q$

$\sim p$

$\therefore q$

Either Socrates is mortal or Socrates is not a human.
Socrates is not mortal,
Therefore Socrates is not a human.

Hypothetical Syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

All humans are animals.
All animals are mortal,
Therefore all humans are mortal.

Constructive Dilemma

$(p \rightarrow q)(r \rightarrow s)$
$p \vee r$
$\therefore q \vee s$

All dogs go to heaven and all cats have YouTube channels.
Abby is a dog or Dorothy is a cat,
Therefore Abby will go to heaven or Dorothy has a YouTube channel.

Absorption

$p \rightarrow q$
$\therefore p \rightarrow pq$

If Abby is a dog then Abby will go to heaven when she dies.

Therefore, if Abby is a dog, then Abby is a dog and Abby will go to heaven when she dies.

Simplification

pq
∴p

Abby is a dog and Abby will go to heaven when she dies.

Therefore Abby is a dog.

Conjunction

p
q
∴pq

Abby is a dog.

Abby will go to heaven when she dies.

Therefore Abby is a dog and Abby will go to heaven when she dies.

Addition

p
$\therefore p \vee q$

Abby is a dog.

Therefore Abby is a dog or humans live on Mars.

Example

Show the justification for the following argument.

$p \to q$

$r \to \sim q$

$\therefore p \to \sim q$

$p \to q$ $r \to \sim q$	

Example

Show the justification for the following argument.

p→q

r→~q

∴p→~q

p→q r→~q	
~r∨~q	Material Implication
~q∨~r	Commutativity
q→~r	Material Implication
(p→q)(q→~r)	Conjunction
∴p→~r	Hypothetical Syllogism

Validity

An argument is **valid** if there is no way for the premises to be true and the conclusion false.

An argument is **invalid** if the premise(s) can be true and the conclusion false.

Validity

An argument is **valid** if there is no way for the premises to be true and the conclusion false.

An argument is **invalid** if the premise(s) can be true and the conclusion false.

Only **valid** arguments can be used as logical proofs.

Invalid Argument Example

$p \rightarrow q$

$\sim p$

$\therefore \sim q$

All humans are mortal.

Socrates is not a human,

Therefore Socrates is not mortal

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

p → q
~p
∴ ~q

Eliminate all rows that would make the premises false.

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

p → q

~p

∴ ~q

Identify a case where the premises are true and the conclusion false.

p: false, q:true

p → q is true

~p is true

~q is false

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

$p \rightarrow q$
$\sim p$
$\therefore \sim q$

It is possible for the premises to be true and the conclusion false, so this is an **invalid** argument.

Invalid Arguments Example

If p then q
q
Therefore p

All humans are mortal.
Socrates is mortal,
Therefore Socrates is a human.

Invalid Argument Example

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

If p then q

q

Therefore p

It is possible for the premises to be true and the conclusion false, so this is an **invalid** argument.

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Invalid Arguments Example

$p \rightarrow q$ $r \rightarrow s$ $p \vee s$ $\therefore q \vee r$	Find a set of truth value assignments where $q \vee r$ is false and the premises are all true.

Invalid Arguments Example

p→q r→s p∨s ∴q∨r	Find a set of truth value assignments where q∨r is false and the premises are all true.
~(q∨r)	Assumption
~q~r	De Morgan’s
~p	Modus Tollens
s	Disjunctive Syllogism
p:False, q:False, r:False, s:True	
∴The argument is invalid	

Invalid Argument Homework Example

- If Deb is smart and works hard, then she will get a good review and will get promoted.
- If Deb is a bit dim-witted but works hard, she will be a valued employee.
- If Deb is a valued employee, she will get promoted.
- If Deb is smart, she will work hard.
- Therefore, Deb will get promoted.

Valid Arguments

All unicorns have white fur.
Lorenzo is a unicorn,
Therefore Lorenzo has white fur.

It doesn’t matter if there are no unicorns. The argument is valid because there is no way for the premises to be true and the conclusion false.

Soundness

An argument is sound if it is valid and the premises are true.

Unsound Arguments

All humans are fish.
Socrates is a human,
Therefore Socrates is a fish.

Unsound Arguments

All humans are fish.
Socrates is a human,
Therefore Socrates is a fish.

Valid!

Unsound Arguments

All humans are fish. ✘
Socrates is a human,
Therefore Socrates is a fish.

Valid!

Unsound.

Unsound Arguments

All unicorns have white fur.
Lorenzo is a Unicorn,
Therefore Lorenzo has white fur.

Unsound Arguments

All unicorns have white fur.
Lorenzo is a Unicorn,
Therefore Lorenzo has white fur.

Valid!

Unsound Arguments

All unicorns have white fur.

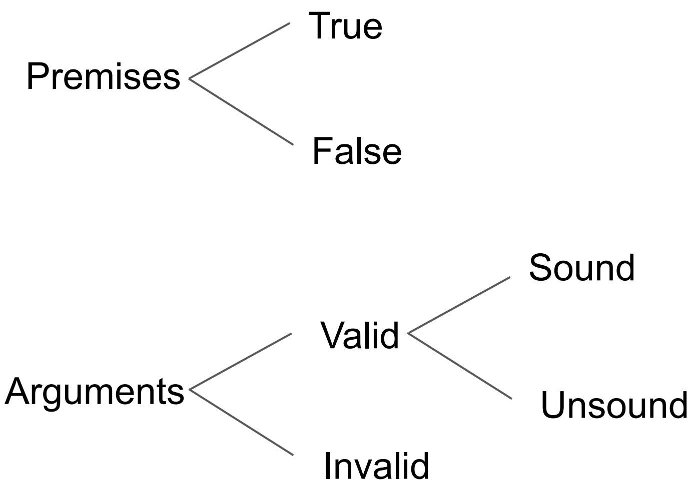
Lorenzo is a Unicorn, ✘

Therefore Lorenzo has white fur.

Valid!

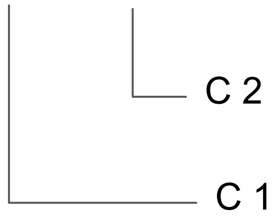
Unsound.

Terminology



Major Premise : defines a relationship between two distinct argument components.

If P then Q

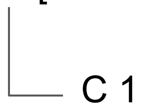


Relationship:

If...then
(Implication)

Minor Premise: provides information about one of the components from the major premise.

P [is true]



Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Proof by Contradiction

Assume the opposite of what you want to prove.
Show a contradiction.

$p \lor q \rightarrow r$
p
$\therefore r$

Use proof by contradiction.

Proof by Contradiction

p∨q→r p ∴r	
~r	Assumption
~(p∨q)	Modus Tollens
~p~q	De Morgan’s
~p	Simplification
p~p	Conjunction
∴r	Proof by Contradiction

Example

Use proof by contradiction to prove the conclusion.

$(p \vee q) \rightarrow rst$

q

$\therefore rs$

Example

(p∨q)→(rs)t q	
~(rs)	Assumption
~[(rs)t]	Constant Evaluation
∴~(p∨q)	M.T.
~p~q	De Morgan’s
~q	Simplification
q~q	Conjunction
∴rs	Proof by Contradiction

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Normal Forms

- Sum of products
- Product of sums

Sum of Products

All propositions are in the form $p_1 \vee p_2 \vee p_3 \vee .... \vee p_n$ Where $p_1 ... p_n$ are in the form $p_1' \wedge p_2' \wedge p_3' \wedge ... p_n'$

Example: Transform $(p \rightarrow q) \wedge r$ into sum of products normal form.

$(p \rightarrow q)r$	
$(\sim p \vee q)r$	Material Implication
$\sim pr \vee qr$	Distribution

Product of Sums

All propositions are in the form $p_1 \wedge p_2 \wedge p_3 \wedge ... p_n$ Where $p_1 ... p_n$ are in the form $p'_1 \vee p'_2 \vee p'_3 \vee .... \vee p'_n$

Example: Transform (p→q)(q→r) into product of sums normal form.

(p→q)(q→r)	
(~p∨q)(q→r)	Material Implication
(~p∨q)(~q→r)	Material Implication

Canonical Forms

A normal form is a canonical form if each term contains **all symbols** being used.

- Two propositions with the same canonical forms are equal.
- Given a logic expression with n distinct symbols in canonical form,

Number of Terms with n symbols	Sum of Products	Product of Sums
0	Contradiction	Tautology
2^n	Tautology	Contradiction
$0 < \text{n-terms} < 2^n$	Contingent	Contingent

Sum of Products Canonical Form

Example: Transform (p→q)∧r into sum of products canonical form.

(p→q)r	
(~p∨q)r	Material Implication
~pr∨qr	Distribution
~pr(q∨~q)∨qr(p∨~p)	Simplification/Excluded Middle
~pqr∨~p~qr∨pqr∨~pqr	Distribution
pqr∨~pqr∨~p~qr	Indentity/Comm.

Sum of Products Canonical Form

Example: Show ~(p⋁r)→~(qr) is a tautology using sum of products canonical form.

~(p⋁r)→~(qr)	
~~(p⋁r)⋁~(qr)	Material Implication
p⋁~q⋁r⋁~r	Double Negation/De Morgan's
p(q⋁~q)(r⋁~r)⋁~q(p⋁~p)(r⋁~r)⋁r(p⋁~p)(q⋁~q)⋁~r(p⋁~p)(q⋁~q)	Excluded Middle
pqr⋁pq~r⋁p~qr⋁p~q~r⋁~pqr⋁~pq~r⋁~p~qr⋁~p~q~r	Distribution/Identity

Sum of Products Canonical Form

Example: Show (p→q)∧p∧~q is a contradiction using sum of products canonical form.

(p→q)p~q	
(~p∨q)p~q	Material Implication
~pp~q∨pq~q	Distribution
null	Non-contradiction/Evaluation

Homework

Construct truth tables for Disjunctive Syllogism, Hypothetical Syllogism, Constructive Dilemma, Absorption, Simplification, Conjunction, and Addition. Convince yourself that these are tautologies.

Homework

If Raphael joins the team then the team will have better fielding.
If Suhil joins the team then the team will have better batting.
Either Raphael or Suhil will join the team.
If the team has better fielding, then Suhil will join the team.
If the team has better batting, then Zhou will join the team.
Therefore either Suhil or Zhou will join the team.

Homework Example

- Let p be the proposition, “Raphael joins the team.”
- Let q be the proposition, “Suhil joins the team.”
- Let r be the proposition, “Zhou joins the team.”
- Let s be the proposition, “The team will have better fielding.”
- Let t be the proposition, “The team will have better batting.”

$p \to s$
$q \to t$
$p \lor q$
$s \to q$
$t \to r$
$\therefore q \lor r$

Homework Example

p→s q→t p∨q s→q t→r ∴q∨r	
(p→s)(q→t) p∨q ∴s∨t	Constructive Dilemma
(s→q)(t→r) s∨t ∴q∨r	Constructive Dilemma

Prepositional Logic

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

Prepositional Logic

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

We cheated!

If Socrates is a human, then Socrates is Mortal.
Socrates is human,
Therefore Socrates is mortal.

Prepositional Logic

Jargon Alert

Attributes/Predicates

All Humans are Mortal.
Socrates is a Human,
Therefore Socrates is Mortal.

Human is an attribute (predicate) of some subset of everything in the universe.
Hx

Mortal is an attribute (predicate) of some subset of everything in the universe.
Mx

Attributes/Predicates

Human is a predicate of some subset of everything in the universe.
Hx

Socrates is Human.
Hs

Mortal is a predicate of some subset of everything in the universe.
Mx

Socrates is Mortal.
Ms

Attributes

H denotes the predicate Human.

Hx

Variable x denotes some unknown instance to which the predicate will be applied.

Hs

Instance s identifies the specific thing in the universe to which H applies. E.g. Socrates is Human.

Conventions.

- Upper-case A-Z letters denote predicates.
- Lower-case a-w denote specific nouns to which the predicate applies.
- Lower-case x,y,z denote variables.
- Upper-case greek letters, e.g. ϕ, denote non-specific predicates.

Prepositions and Propositions

A proposition is a statement to which a truth value can be assigned.

Algebra	Prepositional Logic
Is 3 > x true or false?	Is Hx true or false?
3 > x is not a proposition.	Hx is not a proposition.
Instantiate with x = 2.	Instantiate with x = s where s represents the classical-era philosopher Socrates.
3 > 2 is true.	Hs is true.
3 > 2 is a proposition.	Hs is a proposition.

Prepositions and Propositions

Two ways to convert a preposition into a proposition:

- Instantiation
- Quantification
  - Universal quantification $\forall$
  - Existential quantification $\exists$

Universal Quantification

$\forall(x)Hx$

Everything in the Universe is human.

For all x, x is human.

This is a proposition because we can assign a truth-value to the statement.

Existential Quantification

$\exists(x)Hx$

There is at least one thing in the universe that is human.

Something is human.

This is a proposition because we can assign a truth value to this statement.

Prepositions and Propositions

Statement	Proposition?	Truth-Value
Hx	✗	?
Hs	✓	True
$\forall(x)Hx$	✓	False
$\exists(x)Hx$	✓	True

Closure

Logic functions can be applied to prepositions. The results will also be prepositions.

Let Φx, Ψx be prepositions

- $\sim\Phi x$ is a preposition.
- $\Phi x \vee \Psi x$ is a preposition.
- $\Phi x \wedge \Psi x$ is a preposition.
- $\Phi x \rightarrow \Psi x$ is a preposition.

Quantifiers and Propositions

Quantifiers can be applied to prepositions.

The result is a proposition.

∃(x)~ϕx	There exists and x such that ~ϕx
∃(x)[ϕx∨Ψx]	There exists an x such that ϕx∨Ψx
∃(x)[ϕx∧Ψx]	There exists an x such that ϕx∧Ψx
∃(x)[ϕx→Ψx]	There exists an x such that ϕx→Ψx

Quantifiers and Propositions

Quantifiers can be applied to propositions.

The result is a proposition.

$\forall(x){\sim}\phi x$	For all x, $\sim\phi x$
$\forall(x)[\phi x \vee \Psi x]$	For all x, $\phi x \vee \Psi x$
$\forall(x)[\phi x \wedge \Psi x]$	For all x, $\phi x \wedge \Psi x$
$\forall(x)[\phi x \rightarrow \Psi x]$	For all x, $\phi x \rightarrow \Psi x$

Examples

Convert the following sentences into propositions.

Cats speak French.	
Cats do not speak French.	
Siamese cats are haughty.	
Some cats have fleas.	
Many cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	
Siamese cats are haughty.	
Some cats have fleas.	
Many cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	
Some cats have fleas.	
Many cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	
Many cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	$\exists(x)[Cx \wedge Lx]$
Many cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	$\exists(x)[Cx \land Lx]$
Many cats have fleas.	$\exists(x)[Cx \land Lx]$
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following natural language sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	$\exists(x)[Cx \land Lx]$
Many cats have fleas.	$\exists(x)[Cx \land Lx]$
My cat Dorothy does not have fleas.	$Cd \land \sim Ld$

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Prepositions with Multiple Arguments

Nancy is taller than Marcia.

Let Tx,y mean x is taller than y.

Tn,m

Quantified Prepositions with Multiple Arguments

"Man is the measure of all things."

Let Hx mean x is human.
Let Tx mean x is a thing.
Let Mx,y mean x measures y.

$\forall(x)\forall(y)HxTy \rightarrow Mx,y$

Quantified Prepositions with Multiple Arguments

"Everybody loves somebody sometime."

Let Hx mean x is a person.
Let Tx be a specific point in time.
Let Lx,y,z mean x loves y at time t.

$\forall(x)\exists(y)\exists(z)HxHyTzLx,y,z$

Mostly out of scope for this class.

Quantifiers and Negation

We have different rules for using negation in conjunction with quantifiers.

Let Mx represent the predicate, “x is mortal.”

Mx	x is mortal.
~Mx	x is not mortal.
∀(x)Mx	Everything is mortal.
~∀(x)Mx	Not everything is mortal.
∀(x)~Mx	Nothing is mortal.

Quantifiers and Negation

We have different rules for using negation in conjunction with quantifiers.
Let Mx represent the predicate, “x is mortal.”

Mx	x is mortal.
~Mx	x is not mortal.
∃(x)Mx	At least one thing is mortal.
~∃(x)Mx	Nothing is mortal.
∃(x)~Mx	At least one thing is not mortal.

Contradictories

Quantified Proposition	Contradictory
$\forall (x)\phi x$	$\sim\forall (x)\phi x$
$\exists (x)\phi x$	$\sim\exists (x)\phi x$

Contradictories

Quantified Proposition	Contradictory
∀(x)ϕx	~~~∀(x)ϕx~~ ∃(x)~ϕx
∃(x)ϕx	~~~∃(x)ϕx~~ ∀(x)~ϕx

Contradictories

Quantified Proposition	
$\forall(x)\phi x$	All Humans are Mortal
$\exists(x)\phi x$	At least one human is mortal.

Contradictories

Quantified Proposition		Contradictory	
∀(x)ϕx	All Humans are Mortal	~∀(x)ϕx ∃(x)~ϕx	At least one human is not mortal.
∃(x)ϕx	At least one human is mortal.	~∃(x)ϕx ∀(x)~ϕx	All Humans are not mortal.

Contradictories

Quantified Proposition		Contradictory	
$\forall(x)\phi x$	All Humans are Mortal	~~$\sim\forall(x)\phi x$~~ $\exists(x)\sim\phi x$	At least one human is not mortal.
$\exists(x)\phi x$	At least one human is mortal.	~~$\sim\exists(x)\phi x$~~ $\forall(x)\sim\phi x$	All Humans are not mortal.

Contradictory Examples

Using contradictories, modify each proposition to start with a quantifier rather than a negation.

~∀(x)[Ax→Bx]	
~∀(x)[~Ax~BX]	
~∃(x)[Ax→~Bx]	

Contradictory Examples

Using contradictories, modify each proposition to start with a quantifier rather than a negation.

~∀(x)[Ax→Bx]	∃(x)Ax~Bx
~∀(x)[~Ax~BX]	∃(x)[Ax∨Bx]
~∃(x)[Ax→~Bx]	∀(x)AxBx

Contraries

$\forall(x)\phi x$ and $\forall(x){\sim}\phi x$ are contraries.

Example:
Let Hx represent the predicate, “x is human.”

$\forall(x)Hx$ is false because not everything is human.
$\forall(x){\sim}Hx$ is false because some things are human.

Contraries can both be false, but can’t both be true.

Subcontraries

$\exists(x)\phi x$ and $\exists(x)\sim\phi x$ are subcontraries.

Example:
Let Hx represent the predicate, “The thing x is human.”

$\exists(x)Hx$ is true because at least one thing is human.
$\exists(x)\sim Hx$ is true because at least one thing is not human.

Subcontraries can both be true, but can’t both be false.

Examples

Construct the contrary and contradictory propositions.

Proposition	All politicians are honest.	$\forall(x)[Px \rightarrow Hx]$
Contrary	All politicians are not honest.	$\forall(x){\sim}[Px \rightarrow Hx]$ $\forall(x)Px{\sim}Hx$
Contradictory	Not all politicians are honest. Some politicians are dishonest. There is at least one dishonest politician.	$\exists(x)[Px{\sim}Hx]$

Example

Translate the statement into a quantified predicate. Define its contrary and its contradictory, and then translate the results back into natural language.

All cats are arrogant.

Example

All cats are arrogant.

Let Cx represent, ‘x is a cat.’
Let Ax represent, ‘x is arrogant.’

∀(x)[Cx→Ax]	∀(x)~[Cx→Ax] ∀(x)[Cx~Ax]	~∀(x)[Cx→Ax] ∃(x)[Cx~Ax]
For all things x, if x is a cat then x is arrogant.	Nothing is both a cat and arrogant.	There is at least one thing that is both a cat and not arrogant.

Example

Translate the statement into a quantified predicate. Define its subcontrary and its contradictory, and then translate the results back into natural language.

Some cats are arrogant.

Example

Some cats are arrogant.

Let Cx represent, ‘x is a cat.’
Let Ax represent, ‘x is arrogant.’

∃(x)CxAx	∃(x)~[CxAx]	~∃(x)[CxAx] ∀(x)~[CxAx] ∀(x)[Cx→~Ax]
At least one thing that is a cat is also arrogant.	There is at least one thing that is not both a cat and arrogant.	All cats are not arrogant.

Quantifier Rules of Inference

- Universal Instantiation
- Universal Quantification
- Existential Instantiation
- Existential Quantification

Universal Instantiation

Any universally quantified premise must be true for all instances of the quantified variable.

Rainy Days and Mondays always get me down.
Today is rainy,
Therefore today I am down.

$\forall(x)[(Rx \lor Mx) \rightarrow Ds]$

If x is a rainy day or x is a Monday, then Shaw is down.

Universal Instantiation

$\forall(x)[(Rx \lor Mx) \rightarrow Ds]$

Let t be today and assume it is rainy. Instantiate the universally quantified proposition with the instance t.

$\therefore (Rt \lor Mt) \rightarrow Ds$
$\therefore Ds$

Because $(Rx \lor Mx) \rightarrow Ds$ is true for all instances of x, it must be true for the instance of x that is today, represented by the symbol t.

Universal Instantiation: Substitute an ***instance*** of x for the quantified x.

Example

Assume all roads lead to Rome.
Assume Burns Lane is a road.

∀(x)[Rx→Lx] Rb	
Rb→Lb	Universal Instantiation
∴Lb	Modus Ponens

Therefore, Burns Lane leads to Rome.

Universal Generalization

If a proposition is true for all possible instances, then it can be universally generalized and quantified.

1. Instantiate universal quantifiers with a *general* rather than a *specific* instance.
2. Deduce some other preposition from the result.
3. Generalize to all instances using **Universal Generalization**.

ϕy	Where y is **any** possible instance.
$\therefore \forall (x) \phi x$	Where x is **every** possible instance.

Universal Generalization

Example:

Assume all roads lead to Rome.
Assume all highways are roads.
Prove all highways lead to Rome.

$\forall(x)[Rx \rightarrow Lx]$
$\forall(x)[Hx \rightarrow Rx]$

Prove: $\forall(x)[Hx \rightarrow Lx]$

Universal Generalization

$\forall(x)[Rx \rightarrow Lx]$
$\forall(x)[Hx \rightarrow Rx]$

Instantiate with a *general* rather than a *specific* variable.

Let y be **any** Highway.

Universal Generalization

∀(x)[Rx→Lx] ∀(x)[Hx→Rx] Hy	Assume y is any highway
Hy→Ry	Universal Instantiation
∴Ry	Modus Ponens
Ry→Ly	Universal Instantiation
∴Hy→Ly	Hypothetical Syllogism
∴∀(x)Hx→Lx	Universal Generalization

Existential Instantiation

An existential quantifier implies an instance.

$\exists(x)\phi x$

$\therefore \phi z$

Be sure z is not used in any other context within the proof.

Existential Generalization

If a preposition is true, then it can be existentially quantified.

ϕz

$\therefore \exists (x)\phi x$

Existential Instantiation/Generalization

Example:

All humans are mortal.
Humans exist,
Therefore mortal beings exist.

$\forall(x)[Hx \rightarrow Mx]$
$\exists(x)Hx$
Prove: $\exists(x)Mx$

Existential Instantiation/Generalization

∀(x)[Hx→Mx] ∃(x)Hx	Assume a is a human
Ha	Existential Instantiation
Ha→Ma	Universal Instantiation
∴Ma	Modus Ponens
∴ ∃(x)Mx	Existential Generalization

Existential Instantiation Common Mistakes

Existentially quantified premises instanted with general instances must be done in multiple steps to insure **all instances are different**.

Example:

Some people eat meat
Some lions eat meat
Therefore some people are lions?

Existential Instantiation Common Mistakes

∃ (x)PxMx ∃ (x)LxMx	
PaMa LaMa	Existential Instantiation
Pa	Simplification
La	Simplification
PaLa	Conjunction
∴ ∃ (x)PxLx	Existential Generalization

Existential Instantiation Common Mistakes

∃(x)PxMx ∃(x)LxMx	
PaMa LaMa	~~Existential Instantiation~~
Pa	Simplification
La	Simplification
PaLa	Conjunction
∴ ∃(x)PxLx	Existential Generalization

Existential Instantiation Common Mistakes

∃(x)PxMx ∃(x)LxMx	
PaMa	Existential Instantiation
LbMb	Existential Instantiation

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

Let Px represent x is a person.
Let Cx represent x exercises self-control.
Let Tx represent x keeps their temper.
Let Jx represent x is a judge.

Examples

No one, who exercises self-control, fails to keep his temper.

Some judges lose their tempers.

∀(x)PxCx→Tx

∃(x)PxJx~Tx

Let j represent a judge who loses his temper.

Examples

$\forall(x)PxCx \rightarrow Tx$ $\exists(x)PxJx{\sim}Tx$	Assume Judy is the judge in question.
$PjJj{\sim}Tj$	Existential Instantiation
${\sim}Tj$	Simplification
$PjCj \rightarrow Tj$	Universal Instantiation
$\therefore {\sim}(PjCj)$	Modus Tollens
${\sim}Pj \vee {\sim}Cj$	De Morgan’s
Pj	Simplification
$\therefore {\sim}Cj$	Disjunctive Syllogism
$PjJj{\sim}Cj$	Conjunction
$\therefore \exists(x)PxJx{\sim}Cx$	Existential Generalization

Examples

∀(x)PxCx→Tx ∃(x)PxJx~Tx	Assume Judy is the judge in question.
PjJj~Tj	Existential Instantiation
~Tj	Simplification
PjCj→Tj	Universal Instantiation
∴~(PjCj)	Modus Tollens
~Pj∨~Cj	De Morgan’s
Pj	Simplification
∴~Cj	Disjunctive Syllogism
PjJj~Cj	Conjunction
∴∃(x)PxJx~Cx	Existential Generalization

Conclusion:
Some judges
lack self-control.

Example

“All diligent students are successful.
All ignorant students are unsuccessful.”

What can you conclude about diligent students?
What can you conclude about ignorant students?

Example

“All diligent students are successful.
All ignorant students are unsuccessful.”

Let Tx represent x is a student.
Let Dx represent x is diligent.
Let Ix represent x is ignorant.
Let Sx represent x is successful.

$\forall(x)TxDx \rightarrow Sx$
$\forall(x)TxIx \rightarrow \sim Sx$

Example

∀(x)TxDx→Sx
∀(x)TxIx→~Sx

Q: What can you conclude about diligent students?

A: They are not ignorant.

∀(x)TxDx→Sx ∀(x)TxIx→~Sx TyDy	Assume y is a diligent student.
TyDy→Sy	Universal Instantiation
∴Sy	Modus Ponens
TyIy→~Sy	Universal Instantiation
∴~(TyIy)	Modus Tollens
~Ty∨~Iy	De Morgan's
Ty	Simplification
~Iy	Disjunctive Syllogism
Dy~Iy	Conjunction
∴∀(x)Dx~Ix	Universal Instantiation

Example

∀(x)TxDx→Sx
∀(x)TxIx→~Sx

Q: What can you conclude about ignorant students?

A: They are not diligent.

∀(x)TxDx→Sx ∀(x)TxIx→~Sx TyIy	Assume y is an ignorant student.
TyIy→~Sy	Universal Instantiation
∴~Sy	Modus Ponens
TyDy→Sy	Universal Instantiation
∴~(TyDy)	Modus Tollens
~Ty∨~Dy	De Morgan’s
Ty	Simplification
~Dy	Disjunctive Syllogism
~DyIy	Conjunction
∴∀(x)~DxIx	Universal Instantiation

Example

Some pigs are wild.
All pigs are fat.

Let Px mean x is a pig.
Let Wx mean x is wild.
Let Fx mean x is fat.

Example

∃(x)PxWx
∀(x)Px→Fx

∃(x)PxWx ∀(x)Px→Fx	
PyWy	Existential Instantiation
Py	Simplification
Py→Fy	Universal Instantiation
∴Fy	Modus Ponens
Wy	Simplification
WyFy	Conjunction
∴∃(x)FxWx	Existential Quantification

Example

∃(x)PxWx
∀(x)Px→Fx

Best practice, don't use a generalized variable for Existential Instantiation.

∃(x)PxWx ∀(x)Px→Fx	Assume Charlotte is the wild pig in question.
PcWc	Existential Instantiation
Pc	Simplification
Pc→Fc	Universal Instantiation
∴Fc	Modus Ponens
Wc	Simplification
WcFc	Conjunction
∃(x)FxWx	Existential Quantification

Example

Babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

What can you say about babies?

Bx means x is a baby.
Dx means x is despised.
Ix means x is illogical.
Mx means x can manage a crocodile.

Example

∀(x)[Bx→Ix] ∀(x)[Ix→Dx] ∀(x)[Mx→~Dx] By	Let By be any baby
By→Iy	Universal Instantiation
∴Iy	Modus Ponens
Iy→Dy	Universal Instantiation
∴Dy	Modus Ponens
My→~Dy	Universal Instantiation
∴~My	Modus Tollens
By~My	Conjunction
∴∀(x)Bx~Mx	Babies can’t manage Crocodiles.

Example

1. All writers who understand human nature are clever;
2. No one is a true poet unless he can stir the hearts of men;
3. Shakespeare wrote "Hamlet";
4. No writer, who does not understand human nature, can stir the hearts of men;
5. None but a true poet could have written "Hamlet."

Example

Let Wx mean x is a writer.

Let Ux mean x understands human nature.

Let Cx mean x is clever.

Let Px mean x is a poet.

Let Hx mean x wrote “Hamlet.”

Let Sx mean x stirs the hearts of men.

Example

1. ∀(x)WxUx→Cx
2. ~∃(x)Px~Sx
3. WsHs
4. ~∃(x)Wx~UxSx
5. ~∃(x)Hx~Px

Example

1. ∀(x)WxUx→Cx
2. ∀(x)~[Px~Sx]
3. WsHs
4. ∀(x)~[Wx~UxSx]
5. ∀(x)~[Hx~Px]

Example

∀(x)WxUx→Cx ∀(x)~[Px~Sx] ∀(x)~[Wx~UxSx] ∀(x)~[Hx~Px] WsHs	
Hs	Simp.
~[Hs~Ps]	U.I.
~Hs∨Ps	De.M.
∴Ps	Conj. Syl.
~[Ps~Ss]	U.I.

~Ps∨Ss	De.M.
∴Ss	Conj. Syl.
~[Ws~UsSs]	U.I.
~Ws∨Us∨~Ss	De. M.
∴Us	Conj. Syl.
WsUs	Conj.
WsUs→Cs	U.I.
∴Cs	M.P.

Example

1. Every idea of mine, that cannot be expressed as a Syllogism, is really ridiculous;
2. None of my ideas about Bath-buns are worth writing down;
3. No idea of mine, that fails to come true, can be expressed as a Syllogism;
4. I never have any really ridiculous ideas, that I do not at once refer to my solicitor;
5. My dreams are all about Bath-buns;
6. I never refer to any idea of mine to my solicitor, unless it is worth writing down.

Example

Let Sx represent x is an **idea of mine** that can be expressed as a syllogism.
Let Bx represent x is an **idea of mine** about Bath-buns.
Let Wx represent x is an **idea of mine** worth writing down.
Let Tx represent x is an **idea of mine** that comes true.
Let Rx represent x is an **idea of mine** that is ridiculous.
Let Lx represent x is an **idea of mine** referred to my solicitor.
Let Dx represent x is an **idea of mine** that is a dream.

Example

1. $\forall(x)\sim Sx \rightarrow Rx$
2. $\forall(x)Bx \rightarrow \sim Wx$
3. $\forall(x)\sim Tx \rightarrow \sim Sx$
4. $\forall(x)Rx \rightarrow Lx$
5. $\forall(x)Dx \rightarrow Bx$
6. $\forall(x)Lx \rightarrow Wx$

Example

∀(x)~Sx→Rx ∀(x)Bx→~Wx ∀(x)~Tx→~Sx ∀(x)Rx→Lx ∀(x)Dx→Bx ∀(x)Lx→Wx Dy	Let y represent any dream of mine.
Dy→By	U.I.
∴By	M.P.
By→~Wy	UI
∴~Wy	M.P.

Ly→Wy	U.I.
∴~Ly	M.T.
Ry→Ly	U.I.
∴~Ry	M.T.
~Sy→Ry	U.I.
∴Sy	M.T.
~Ty→~Sy	U.I.
∴~~Ty	M.T.
Ty	D.N.
DyTy	Conj.
∴∀(x)DxTx	U.G.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

Example

Let Fx represent x is a fig.

Let Gx represent x is a grape.

Let Hx represent x is healthful.

Let Ix represent x is illaudable.

Let Jx represent x is jejune.

Let Kx represent x is knurly.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx
∴∃(x)FxIx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx ∴∃(x)FxIx?	Assume f is a fig that is knurly.
Ff~Kf	E.I.
Ff	Simp.
(Ff∨Gf)→Hx	U.I.
∴Hf	M.P.
Hf→~If~Jf	U.I.
∴~If~Jf	M.P.
~If	Simp.
∴∃(x)FxKx~Ix	

This is not at all what we set out to prove.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx
∴∃(x)FxIx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx ∴∃(x)FxIx? Fy	Assume y is any fig.
(Fy∨Gy)→Hy	U.I.
∴Hy	M.P.
Hy→~Iy~Jx	U.I.
∴~Iy~Jy	M.P.
~Iy	U.I.
Fy~Iy	Conj
∴∀(x)Fx~Ix	Simp.
~∴∃(x)FxIx	Refute by contradiction?

This is not at all what we set out to prove.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx	Assume g is jejune and knurly grape.
GgJgKg	E.I.
Jg	Simp.
(Fg∨Gg)→Hg	U.I.
∴Hg	M.P.
Hg→~Ig~Jg	U.I.
∴~Ig~Jg	M.P.
~Jg	Simp.
Jg~Jg	

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx	Assume g is jejune and knurly grape.
GgJgKg	E.I.
Jg	Simp.
(Fg∨Gg)→Hg	U.I.
∴Hg	M.P.
Hg→~Ig~Jg	U.I.
∴~Ig~Jg	M.P.
~Jg	Simp.
Jg~Jg	

The problem is ill-formed. The premises cannot all be true.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. Therefore, some figs are illaudable.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- ~~Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.~~
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. Therefore, some figs are illaudable.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- ~~Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.~~
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. ~~Therefore, some figs are illaudable.~~

Explain the transformation and keep the original for reference.

Conclusion

Formal logic is a set of techniques and rules of inference used to assign logic values to propositions.

If a statement can’t be assigned a truth value, then the statement is not subject to formal logic.

- Identify the prepositions.
- Identify the quantifiers.
- Transform language into propositions.
- Verify validity.
- Use function/replacement rules/rules of inference to draw conclusions.

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