Formal Logic
AKA
Symbolic Logic

We will be working problems, so please have paper and a writing instrument on hand.

Course Outline

- Day 1
  - Propositional Logic Part 1
- Day 2
  - Propositional Logic Part 2
  - Prepositional Logic Part 1
- Day 3
  - Prepositional Logic Part 2
- Day 4
  - Prepositional Logic Part 3

Scope

Informal Logic

Set of techniques used to evaluate arguments made in everyday language.

Formal Logic

Set of formulae used to assign truth values to symbolic equations.

Scope

Informal Logic

Set of techniques used to evaluate arguments made in everyday language.

Formal Logic

Rules of inference and replacement used to assign truth values to symbolic equations.

Why Bother?

- Eliminate ambiguity
- Remove bias
- Nullify uncertainty

Transform messy problems of language into clear, elegant, and unambiguous math problems.

Abductive, Inductive, and Deductive Reasoning

Abductive

Start with a concrete instance.
Draw a conclusion based on the best explanation.

My lettuce plants nearly always die in January after a cold night.
I hypothesize frosts in January kill my plants.

Abductive, Inductive, and Deductive Reasoning

Inductive

Start with an hypothesis.
Find supporting evidence.
Find contradictory evidence.
Generalize.

Frost kills lettuce plants in January.
True in 1970 - 2002, and 2004 - 2021.
False in 2003, 2022. (Yes, some lettuce survived this year.)
For practical purposes, assume lettuce will not survive through January.

Abductive, Inductive, and Deductive Reasoning

Deductive

Start with what you know.
Draw a conclusion.

Frost usually kills lettuce in January.
Lettuce takes 60 days to produce a crop.
Therefore, for reliable results, plant lettuce no later than October.

Abductive, Inductive, and Deductive Reasoning

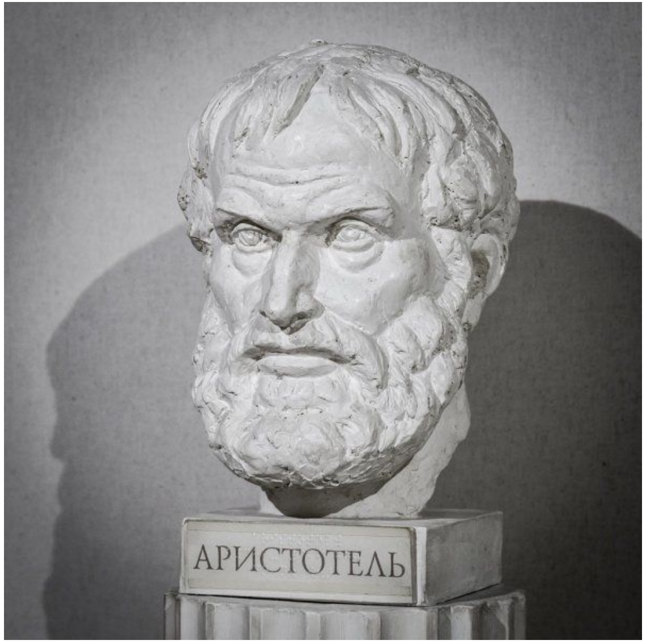
Deductive

Start with what you know.
Draw a conclusion.

Frost usually kills lettuce in January.
Lettuce takes 60 days to produce a crop.
Therefore, for reliable results, plant lettuce no later than October.

The domain of this course is **deductive logic**.

Where did it all Begin?



Aristotle, 384 - 322 B.C.E.

Socrates -> Plato -> Aristotle

- Three laws of thought.
  - Excluded Middle
  - Non-Contradiction
  - Identity
- The syllogism as the basis of deductive logic.

The Proposition

A statement to which a truth-value of true or false can be assigned.

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All humans are mortal.

All dogs are human.

What is my dog’s name?

Brutalist Architecture is ugly.

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What is my dog’s name? ✗

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The Proposition

A statement to which a truth-value of true or false can be assigned.

All humans are mortal. ✓

All dogs are human. ✓

What is my dog’s name? ✗

Brutalist Architecture is ugly. ✗

Symbols

Let p represent the proposition, “All humans are mortal.”

Let q represent the proposition, “All dogs are human.”

Use p, q, … in place of the full proposition when defining logic equations.

Truth Values

Let p be a proposition.

Then p **must** evaluate to **exactly one** element from the set {True, False}.

Excluded Middle

In math-speak, propositions are the domain and {True, False} is the range, where the function is one-to-one.

Truth Values

p
True
False

Truth Tables

Peirce (1839 - 1914) or Wittgenstein (1889 - 1951)

Truth tables list all possible values of propositions and derived propositions.

Given n propositions, there are 2^n rows in the corresponding truth table.

p
True
False

p	q
True	True
True	False
False	True
False	False

Logical Functions

Not, Negation

Let p be a proposition.

~p, !p, ¬p, Not(p), $\overline{p}$ all denote the negation function applied to p.

p	~p
True	False
False	True

Logical Functions

Or, Disjunction

Let p, q be propositions

p+q, p∨q, p or q, OR(p,q) denote disjunction applied to p, q.

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Logical Functions

p∨q is False **only** when both p and q are false.

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Linguistic versus Logical OR

"Are you going today, **or** tomorrow?"

Linguistic versus Logical OR

“Are you going today, **or** tomorrow?”

“Today.” or perhaps, “Tomorrow.” Maybe even, “Neither.” But probably not “yes” or “no.”

Linguistic versus Logical OR

"Are you going today, **or** tomorrow?"

"Today." or perhaps, "Tomorrow." Maybe even, "Neither." But probably not "yes" or "no."

This is not a proposition. For the sake of your interpersonal relationships, do not answer, "yes."

Excluded Middle

A proposition must have exactly one value from the set {True, False}.

p	~p	p ∨ ~p
True	False	True
False	True	True

Excluded Middle

A proposition must have exactly one value from the set {True, False}.

p	~p	p ∨ ~p
True	False	True
False	True	True

A **tautology** is a compound logic statement that always evaluates to True.

Logical Functions

And, Conjunction

Let p, q be propositions

pq, $p \cdot q$, $p \land q$, p AND q, AND(p,q) denote conjunction applied to p, q.

p	q	$p \land q$
True	True	True
True	False	False
False	True	False
False	False	False

Logical Functions

$p \land q$ is true only when both p and q are true.

p	q	$p \land q$
True	True	True
True	False	False
False	True	False
False	False	False

Logical Functions

$p \land q$ is true only when both p and q are true.

p	q	$p \land q$
True	True	True
True	False	False
False	True	False
False	False	False

The compound statement $p \land q$ is sometimes True and sometimes False. This is called a **contingency**.

Linguistic Versus Logical AND

“Rainy days and Mondays always get me down.”

Linguistic Versus Logical AND

"Rainy days and Mondays always get me down."

Rainy days get me down.
Mondays get me down.
Rainy Mondays get me down.

Linguistic 'and' often means logical OR.

Non-Contradiction

p and ~p can never both be true.

p	~p	$p \land {\sim}p$
True	False	False
False	True	False

Non-Contradiction

p and ~p can never both be true.

p	~p	$p \wedge \sim p$
True	False	False
False	True	False

A **contradiction** is a compound logic statement that always evaluates to False.

Logical Functions

Implication, If/Then

Let p, q be propositions

$p \rightarrow q$, $p \Rightarrow q$, $p \supset q$, IF p THEN q, denote implication applied to p, q.

p	q	$p \rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Note: p is called the premise and q the conclusion.

Logical Functions

p→q is false **only** when the premise is true and the conclusion false.

p	q	p→q
True	True	True
True	False	False
False	True	True
False	False	True

Logical Functions

$p \rightarrow q$ is false only when the premise is true and the conclusion false.

p	q	$p \rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

“False implies anything.”

Linguistic Versus Logical IF/Then

If I buy this car then I will overdraw my checking account.

The statement is linguistically meaningful only when the premise (If I buy this car) is true.

Similar to a logical AND rather than a logical IF/THEN.

Identity

$p \rightarrow p$ or $p = p$.

p	$p \rightarrow p$
True	True
False	True

$p \rightarrow p$ is a tautology.

Closure

The proposition domain is closed under ~,∨,∧,→.

That is,

If p is a proposition, then ~p is a proposition.
If p,q are propositions, then p∨q is a proposition.
If p,q are propositions, then p∧q is a proposition.
If p,q are propositions, then p→q is a proposition.

Examples

In the following examples, convert the text into symbolic logic statements, then assign a truth value to each resulting statement.

Examples

In the following examples, convert the text into symbolic logic statements, then assign a truth value to each resulting statement.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Examples

Step 1: define symbolic representations.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Let p represent the statement, “My cat speaks French.”

Let q represent the statement, “Williamsburg is the capital of Virginia.”

Examples

Step 2: construct the logic statement.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Let p represent the statement, “My cat speaks French.”

Let q represent the statement, “Williamsburg is the capital of Virginia.”

$\sim p \vee q$

Examples

Step 3: Evaluate the statement given the propositions' truth values.

Either my cat does not speak French or Williamsburg is the capital of Virginia.

Let p represent the statement, "My cat speaks French."
p = False

Let q represent the statement, "Williamsburg is the capital of Virginia."
q = False

$\sim p \lor q = \sim(\text{False}) \lor \text{False} = \text{True} \lor \text{False} =$ **True**

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

Let p be, “My cat speaks French.”

Let q be, “Williamsburg is the capital of Virginia.”

$p \rightarrow q$

Examples

If my cat speaks French, then Williamsburg is the capital of Virginia.

p = False

q = False

False→False = **True**

Exercises

Convert the following statements into symbolic propositions.

1. “Baby’s in black and I’m feeling blue.”
2. “If there’s something strange in your neighborhood, who(m) you gonna call? Ghost Busters!”
3. “We saw no plowed fields, very few villages, no trees or grass or vegetation of any kind…”
4. Polyester knock-offs are ugly and I won’t buy them.
5. Either I will make a trip to the grocery store, or we will be having beans and rice for supper.

Solutions

“Baby’s in black and I’m feeling blue.”

Let p represent “Baby is in black.”

Let q represent “I’m feeling blue.”

$p \land q$

“Baby is in black and I’m feeling blue.”

Solutions

“If there’s something strange in your neighborhood, who(m) you gonna call? Ghost Busters!”

Let p represent “There’s something strange in my neighborhood.”

Let q represent, “I will call Ghost Busters.”

$p \rightarrow q$

“If there is something strange in my neighborhood, then I will call Ghost Busters.”

Solutions

"We saw no plowed fields, very few villages, no trees or grass or vegetation of any kind…"

Let p represent, "We saw plowed fields."
Let q represent, "We saw few villages."
Let r represent, "We saw trees."
Let s represent, "We saw grass."
Let t represent, "We saw vegetation."

$\sim p \wedge q \wedge \sim r \wedge \sim s \wedge \sim t == \sim p \wedge q \wedge \sim(r \vee s \vee t)$

We did not see plowed fields and we saw few villages and we did not see trees and we did not see grass and we did not see vegetation.

Solutions

Polyester knock-offs are ugly and I won’t buy them.

Let p represent, “Polyester knock-offs are ugly.”

Let q represent, “I will buy polyester knock-offs.”

Solutions

Polyester knock-offs are ugly and I won't buy them.

~~Let p represent, "Polyester knock-offs are ugly."~~

Let q represent, "I will buy polyester knock-offs."

Trick question. P is not a proposition because it cannot be assigned a value of True or False.

Solutions

Either I will make a trip to the grocery store, or we will be having beans and rice for supper.

Let p represent, “I will go to the grocery store.”

Let q represent, “We will be having beans and rice for supper.”

$p \lor q$

I will go to the store or we will be having beans and rice for supper.

$\sim p \rightarrow q$

If I do not go to the store then we will be having beans and rice for supper.

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Solutions

What is the relationship between proposition $p \lor q$ and proposition $\sim p \to q$?

p	q
True	True
True	False
False	True
False	False

Solutions

What is the relationship between proposition $p \vee q$ and proposition $\sim p \rightarrow q$?

p	q	$p \vee q$
True	True	True
True	False	True
False	True	True
False	False	False

Solutions

What is the relationship between proposition $p \vee q$ and proposition $\sim p \rightarrow q$?

p	q	$\sim p$	$p \vee q$	$\sim p \rightarrow q$
True	True	False	True	True
True	False	False	True	True
False	True	True	True	True
False	False	True	False	False

Solutions

What is the relationship between proposition p ∨ q and proposition ~p → q?

p	q	~p	p ∨ q	~p → q
True	True	False	True	True
True	False	False	True	True
False	True	True	True	True
False	False	True	False	False

These are logically identical propositions.

Exercises

Use Truth Tables to determine if the following sets of propositions are equivalent.

$pq \vee \sim p \sim q$	$p \rightarrow q$
$p \rightarrow q \rightarrow r$	$\sim p \vee r$
$pq \vee \sim pq \vee p \sim q \vee \sim p \sim q$	$p \vee \sim p$
$(p \vee q)(\sim p \vee q)(p \vee \sim q)(\sim p \vee \sim q)$	$p \vee \sim q$

Solutions

Compare pq ⋁ ~p~q and p → q

p	q	~p	~q	pq	~p~q	pq ⋁ ~p~q	p → q
True	True	False	False	True	False	True	True
True	False	False	True	False	False	False	False
False	True	True	False	False	False	False	True
False	False	True	True	False	True	True	True

Solutions

Compare $p \rightarrow q \rightarrow r$ and $\sim p \vee r$

p	q	r	~p	p → q	p → q → r	~p∨r
True	True	True	False	True	True	True
True	True	False	False	True	False	False
True	False	True	False	False	True	True
True	False	False	False	False	True	False
False	True	True	True	True	True	True
False	True	False	True	True	False	True
False	False	True	True	True	True	True
False	False	False	True	True	False	True

Solutions

Compare **pq ∨ ~pq ∨ p~q ∨ ~p~q** and **p ∨ ~p**

p	q	pq	~pq	p~q	~p~q	pq ∨ ~pq ∨ p~q ∨ ~p~q
True	True	True	False	False	False	True
True	False	False	False	True	False	True
False	True	False	True	False	False	True
False	False	False	False	False	True	True

Solutions

Compare (p ∨ q)(~p ∨ q)(p ∨ ~q)(~p ∨ ~q) and p ∨ ~q

p	q	p∨q	~p∨q	p∨~q	~p∨~q	(p ∨ q)(~p ∨ q)...
True	True	True	True	True	False	False
True	False	True	False	True	True	False
False	True	True	True	False	True	False
False	False	False	True	True	True	False

Evaluating Logic Expressions

1. Convert language to symbolic expressions.
2. Assign truth values to each base proposition.
3. Evaluate compound expressions given the order of operations.
4. Assign a truth value to the entire expression.

Order of Operations

When evaluating logic expressions, perform evaluations in this order, from left to right.

1. Parentheses
2. Negation
3. Conjunction
4. Disjunction
5. Implication

Example

“She was going to meet her connection.”
“You get what you need.”
“You always get what you want.”

Let p represent, “She was going to meet her connection.”
Let q represent, “You get what you need.”
Let r represent, “You always get what you want.”

Assume p,q are True and r is False.
Evaluate the following logic formula.

$\sim(pq) \rightarrow (p \lor r)$

Example

Assume p,q are True and r is False.

Evaluate the following logic formula.

$\sim(pq) \rightarrow (p \vee r)$
$\sim True \rightarrow (p \vee r)$
$\sim True \rightarrow True$
$False \rightarrow True$
True

It is true that:

If it is not the case that she was both going to meet her connection and you get what you need, then either she is going to meet her connection or you always get what you want.

Examples

"Jude takes a sad song and makes it better."
"Jude is not afraid."
"Jude has not found 'her'."
"Jude makes it bad."
Let a represent, "Jude takes a sad song and makes it better."
Let b represent, "Jude is not afraid."
Let x represent, "Jude has not found 'her',"
Let y represent, "Jude makes it bad."

Assume a,b are True and x,y are False.
Evaluate the following logic formula.

$$(a \to b) \to \sim(a \vee x \to y)$$

Examples

Assume a,b are True and x,y are False.
Evaluate the following logic formula.

(a→b)→~(a∨x→y)
True→~(a∨x→y)
True→~(True→y)
True→~False
True→True
True

It is true that:

If it is the case that if Jude takes a bad song and makes it better then Jude is not afraid, then it is not the case that if either Jude takes a bad song and makes it better or Jude has not found 'her' then Jude makes it bad.

Examples

Assume a,b are True and x,y are False.

Find the truth value of the following proposition.

$\sim x \vee y \wedge a \rightarrow (b \rightarrow \sim(a \vee x))$

Examples

Assume a,b are True and x,y are False.
Find the truth value of the following proposition.

$\sim x \vee y \wedge a \rightarrow (b \rightarrow \sim(a \vee x))$
$\sim x \vee y \wedge a \rightarrow (b \rightarrow \sim True)$
$\sim x \vee y \wedge a \rightarrow (b \rightarrow False)$
$\sim x \vee y \wedge a \rightarrow False$
$True \vee y \wedge a \rightarrow False$
$True \vee False \rightarrow False$
$True \rightarrow False$
False

Problems

Given the following logic expressions and truth values, find the truth value of each proposition.

1. $p \lor (q{\sim}p)$, p is True and q is False
2. $p \lor (q{\sim}p)$, p is False and q is True
3. $(p{\sim}q)\rightarrow[(p\rightarrow q)\rightarrow{\sim}({\sim}pq)]$, p is True and q is False
4. $(p{\sim}q)\rightarrow[(p\rightarrow q)\rightarrow{\sim}({\sim}pq)]$, p is True and q is True

Solutions

p ∨ (q~p), p = True, q = False
p ∨ (False ∧ ~True)
p ∨ (False ∧ False)
p ∨ False
True ∨ False
True

Solutions

p ∨ (q~p), p = False, q = True
p ∨ (True ∧ ~False)
p ∨ (True ∧ True)
p ∨ True
False ∨ True
True

Solutions

(p~q)→[(p→q)→~(~pq)] p is True and q is False
(p~q)→[(p→q)→~(~True∧q)]
(p~q)→[(p→q)→ ~(False∧False)]
(p~q)→[(p→q)→~False]
(p~q)→[(p→q)→True]
(p~q)→[(True→False)→True]

(p~q)→[False→True]
(p~q)→True
(p∧~False)→True
(True∧True)→True
True→True
True

Solutions

(p~q)→[(p→q)→~(~pq)] p is True and q is True
(p~q)→[(p→q)→~(~True∧q)]
(p~q)→[(p→q)→ ~(False∧True)]
(p~q)→[(p→q)→~False]
(p~q)→[(p→q)→True]
(p~q)→[(True→True)→True]

(p~q)→[True→True]
(p~q)→True
(p∧~True)→True
(True∧False)→True
False→True
True

Replacement or Equivalence Rules

Use these rules to evaluate, simplify, or rearrange propositions.

- Constant Evaluation
- Commutativity
- Associativity
- Distribution
- Double Negation
- Tautology
- Material Implication
- De Morgan’s

Constant Evaluation

Let p be a proposition.

$False \wedge p = False$

$False \rightarrow p = p$

$True \wedge p = p$

$True \vee p = True$

$p \wedge \sim p = False$ (Non-Contradiction)

$p \vee \sim p = True$ (Excluded Middle)

$p \rightarrow p = True$ (Identity)

$False \rightarrow p = True$ (False Implies Anything)

Problems, Revisited

p ∨ (q~p), p is True and q is False

Given the replacement rule
True ∨ x = True

Since p is True, the entire proposition is True

p ∨ (q~p), p is False and q is True

Given the replacement rule
False ∨ x = x

Since p is False, we only need to evaluate q~p

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Examples

Let p = false, q = true, r = true and s = false. Evaluate the following propositions.

1. $p \rightarrow \sim(qr)\sim(sr)pq \vee r$
2. $q \vee \sim(qr)\sim(sr)pq$
3. $\sim(qr)\sim(sr)pq$
4. $q \vee \sim q \vee s$

Solutions

Let p = false, q = true, r = true and s = false. Evaluate the following propositions.

p → ~(qr)~(sr)pq∨r	
False → ~(qr)~(sr)pq∨r	Variable Replacement
True	Constant Evaluation

Solutions

Let p = false, q = true, r = true and s = false. Evaluate the following propositions.

q ⋁ ~(qr)~(sr)pq	
True ⋁ ~(qr)~(sr)pq	Variable Replacement
True	Constant Evaluation

Solutions

Let p = false, q = true, r = true and s = false. Evaluate the following propositions.

~(qr)~(sr)∧p∧q	
~(qr)~(sr)∧False∧q	Variable Replacement
False	Constant Evaluation

Solutions

Let p = false, q = true, r = true and s = false. Evaluate the following propositions.

q ⋁ ~q ⋁ s	
True ⋁ s	Excluded Middle
True	Constant Evaluation

Commutativity

Let p,q be propositions.

Then $p \vee q = q \vee p$

Then $p \wedge q = q \wedge p$

p	q	$p \vee q$	$q \vee p$
True	True	True	True
True	False	True	True
False	True	True	True
False	False	False	False

p	q	$p \wedge q$	$q \wedge p$
True	True	True	True
True	False	False	False
False	True	False	False
False	False	False	False

Commutativity

Let p,q be propositions.

Then p→q ≠ q→p. (I.e. implication is not commutative.)

p	q	p→q	q→p
True	True	True	True
True	False	False	True
False	True	True	False
False	False	True	True

Associativity

Let p,q,r be propositions.

Then $(p \lor q) \lor r = p \lor (q \lor r)$

p	q	r	$p \lor q$	$q \lor r$	$(p \lor q) \lor r$	$p \lor (q \lor r)$
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	True	True	True
True	False	False	True	False	True	True
False	True	True	True	True	True	True
False	True	False	True	True	True	True
False	False	True	False	True	True	True
False	False	False	False	False	False	False

Associativity

Let p,q,r be propositions.

Then $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
True	True	True	True	True	True	True
True	True	False	True	False	False	False
True	False	True	False	False	False	False
True	False	False	False	False	False	False
False	True	True	False	True	False	False
False	True	False	False	False	False	False
False	False	True	False	False	False	False
False	False	False	False	False	False	False

Associativity

Let p,q,r be propositions.

Then $(p \to q) \to r \neq p \to (q \to r)$

(I.e. Implication is not associative.)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \to r$	$p \to (q \to r)$
True	True	True	True	True	True	True
True	True	False	True	False	False	False
True	False	True	False	True	True	True
True	False	False	False	True	True	True
False	True	True	True	True	True	True
False	True	False	True	False	False	True
False	False	True	True	True	True	True
False	False	False	True	True	False	True

Drop the Parentheses

Let p_1, p_2, …, p_n be propositions.

Then $(((p_1 \vee p_2) \vee p_3)...p_n) = p_1 \vee p_2 \vee p_3 ... \vee p_n$

Then $(((p_1 \wedge p_2) \wedge p_3)...p_n) = p_1 \wedge p_2 \wedge p_3 ... \wedge p_n = p_1 p_2 p_3 ... p_n$

Drop the Parentheses

Let p_1, p_2, …, p_n be propositions.

Then $(((p_1 \vee p_2) \vee p_3)...p_n) = p_1 \vee p_2 \vee p_3 ... \vee p_n$

Then $(((p_1 \wedge p_2) \wedge p_3)...p_n) = p_1 \wedge p_2 \wedge p_3 ... \wedge p_n = p_1 p_2 p_3 ... p_n$

This only works with all $\wedge$ or all $\vee$. In a mix, retain the parentheses.

Example

Assume p,q,r are propositions where p is false, q is false, and r is true.

Evaluate $p \land (q \lor r)$ and $p \land q \lor r$.

$p \land (q \lor r)$	$p \land q \lor r$
$p \land$ True	False $\lor\, r$
False $\land$ True	False $\lor$ True
False	True

Distribution

Let p,q,r be propositions.

Then $p(q \vee r) = pq \vee pr$

Distribution

p	q	r	q∨r	p(q∨r)	pq	pr	pq∨pr
True	True	True	True	True	True	True	True
True	True	False	True	True	True	False	True
True	False	True	True	True	False	True	True
True	False	False	False	False	False	False	False
False	True	True	True	False	False	False	False
False	True	False	True	False	False	False	False
False	False	True	True	False	False	False	False
False	False	False	False	False	False	False	False

Distribution

Let p,q,r be propositions.

Then $p \vee (qr) = (p \vee q)(p \vee r)$

Distribution

p	q	r	qr	p∨(qr)	p∨q	p∨r	(p∨q)(p∨q)
True	True	True	True	True	True	True	True
True	True	False	False	True	True	True	True
True	False	True	False	True	True	True	True
True	False	False	False	True	True	True	True
False	True	True	True	True	True	True	True
False	True	False	False	False	True	False	False
False	False	True	False	False	False	True	False
False	False	False	False	False	False	False	False

Example

Rewrite the logic statement to remove the parentheses.

$(p \vee q)(r \vee \sim q)$	
$x(r \vee \sim q)$	Let $x = (p \vee q)$
$rx \vee \sim qx$	Dist.
$r(p \vee q) \vee \sim q(p \vee q)$	Substitution
$rp \vee rq \vee \sim qp \vee \sim qq$	Dist.
$rp \vee rq \vee \sim qp \vee \text{False}$	Non-Contradiction
$rp \vee rq \vee \sim qp$	Const. Eval.

Double Negation

Let p be a proposition.

Then ~~p = p.

p	~p	~~p
True	False	True
False	True	False

Tautology

Let p be a proposition.

Then $p = p \vee p$ and $p = p \wedge p$.

p	$p \vee p$	$p \wedge p$
True	True	True
False	False	False

Material Implication

Let p, q be propositions.

Then $p \rightarrow q = {\sim}p \vee q$

p	q	$p \rightarrow q$	${\sim}p$	${\sim}p \vee q$
True	True	True	False	True
True	False	False	False	False
False	True	True	True	True
False	False	True	True	True

De Morgan's Theorem

De Morgan (1806 - 1871) Professor of Mathematics at University College, London.

- Formalized Mathematical Induction.
- Investigated informal negation rules for inclusion in formal logic.

De Morgan’s Theorem

Let p, q be propositions.

Then $\sim(pq) = \sim p \vee \sim q$, and

$\sim(p \vee q) = \sim p \wedge \sim q$

De Morgan’s Theorem

Let p, q be propositions.

Then ~(pq) = ~p∨~q.

p	q	pq	~(pq)	~p	~q	~p∨~q
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

De Morgan’s Theorem

Let p, q be propositions.

Then ~(p∨q) = ~p∧~q.

p	q	p∨q	~(p∨q)	~p	~q	~p~q
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Example

Remove the parentheses and simplify the following statement.

$\sim(p \vee r) \rightarrow \sim(qr)$	

Example

Remove the parentheses and simplify the following statement.

~(p∨r)→~(qr)	
~~(p∨r)∨~(qr)	Material Implication
(p∨r)∨~(qr)	Double Negation
(p∨r)∨(~q∨~r)	De Morgan's
p∨~q∨(r∨~r)	Comm./Assoc.
p∨~q∨True	Excluded Middle
True	Constant Evaluation
∴~(p∨r)→~(qr) is a tautology	

Example

Determine if the following formula is a contradiction, a tautology, or a contingent statement.

$q \to (p \vee \sim q)$	

Example

Determine if the following formula is a contradiction, a tautology, or a contingent statement.

$q \rightarrow (p \lor \sim q)$	
$\sim q \lor (p \lor \sim q)$	Material Implication
$\sim q \lor (\sim q \lor p)$	Comm.
$(\sim q \lor \sim q) \lor p$	Assoc.
$\sim q \lor p$	Identity

Example

~q∨p

p	q	~q	~q∨p
True	True	False	True
True	False	True	True
False	True	False	False
False	False	True	True

∴q→(p∨~q) is a contingent.

Problems

Remove the parentheses from the following propositions, then simplify if possible.

1. $q(p \vee {\sim}q)$
2. $(p \vee q)(p \vee {\sim}r)r$
3. $(p \vee q) \rightarrow [(p \rightarrow q) \rightarrow {\sim}(q \vee {\sim}q)]$

Solutions

q(p∨~q)	
pq ∨ q~q	Distribution
pq ∨ False	Non-contradiction
pq	Constant Evaluation

Solutions

$(p \lor q)(p \lor \sim r)r$	
$(p \lor q)(pr \lor \sim rr)$	Distribution
$(p \lor q)(pr \lor \text{False})$	Non-contradiction
$(p \lor q)pr$	Constant Evaluation
$ppr \lor pqr$	Distribution
$pr \lor pqr$	Tautology
pr	Truth Table

Show pr ∨ pqr = pr

p	q	r	pr	pqr	pr∨pqr
True	True	True	True	True	True
True	True	False	False	False	False
True	False	True	True	False	True
True	False	False	False	False	False
False	True	True	False	False	False
False	True	False	False	False	False
False	False	True	False	False	False
False	False	False	False	False	False

Solutions

$(p \vee q) \rightarrow [(p \rightarrow q) \rightarrow \sim(q \vee \sim q)]$	
$\sim(p \vee q) \vee [(p \rightarrow q) \rightarrow \sim(q \vee \sim q)]$	Material Implication
$\sim(p \vee q) \vee [(p \rightarrow q) \rightarrow \sim True]$	Excluded Middle
$\sim(p \vee q) \vee [(p \rightarrow q) \rightarrow False]$	Negation
$\sim(p \vee q) \vee [\sim(p \rightarrow q) \vee False]$	M.I.
$\sim(p \vee q) \vee \sim(p \rightarrow q)$	Constant Evaluation
$\sim(p \vee q) \vee \sim(\sim p \vee q)$	M.I.

Solutions

$\sim(p \vee q) \vee \sim(\sim p \vee q)$	
$\sim p \sim q \vee \sim\sim p \sim q$	De.M.
$\sim p \sim q \vee p \sim q$	Double Negation
$\sim q(p \vee \sim p)$	Dist./Comm.
$\sim q \wedge \text{True}$	Excluded Middle
$\sim q$	Constant Evaluation

The Syllogism

“When a Trio of Biliteral Propositions of Relations is such that

(1) all their six Terms are Species of the same Genus,
(2) every two of them contain between them a Pair of codivisional Classes,
(3) the Three Propositions are so related that, if the first two were true, the third would be true,

the trio is called a ‘Syllogism’;”

Lewis Carroll
Symbolic Logic

The Syllogism

> “A syllogism is a form of deductive argument where the conclusion follows from the truth of two (or more) premises.”*

The Syllogism, combined with the Laws of Thought, form the core of Aristotle’s Deductive Logic system.

*https://thedecisionlab.com/reference-guide/philosophy/syllogism

The Syllogism

Major Premise	Relate two propositions and assert the relationship is True.
Minor Premise	Assert the truth of a value or functional relationship of a proposition from the major premise.
Conclusion	Derive the truth value or relationship of the second proposition.

The Syllogism

Major Premise	Relate two propositions and **assert the relationship is True**.
Minor Premise	Assert the truth of a value or functional relationship of a proposition from the major premise.
Conclusion	Derive the truth value or relationship of the second proposition.

Modus Ponens

If p then q p Therefore q	$p \rightarrow q$ p $\therefore q$

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

Modus Ponens

Modus Ponens is a tautology.

p	q	$p \rightarrow q$
True	True	False
True	False	True
False	True	False
False	False	True

Modus Ponens

Modus Ponens is a tautology.

$p \rightarrow q$ is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

Modus Ponens

Modus Ponens is a tautology.

p is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
~~False~~	~~False~~	~~True~~

Modus Ponens

Modus Ponens is a tautology.

q is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
~~False~~	~~False~~	~~True~~

Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

We’ve had 1.3 cm of rain in the past week,
Therefore I don’t need to water my garden.

Example

If we've had at least 1 cm of rain in the past week, I don't need to water my garden.
We've had 1.3 cm of rain in the past week,
Therefore I don't need to water my garden.

Let p be the proposition, "We've had at least 1 cm of rain in the past week."
Let q be the proposition, "I need to water my garden."

$p \rightarrow \sim q$
p
$\therefore \sim q$

Modus Tollens

If p then q Not q Therefore Not p	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

All humans are mortal.
Socrates is not mortal,
Therefore Socrates is not a human.

Modus Tollens

Modus Tollens is a tautology.

p→q is True

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

Modus Tollens

Modus Tollens is a tautology.

q is False (q is not True)

p	q	$p \to q$
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
False	False	True

Modus Tollens

Modus Tollens is a tautology.

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
~~False~~	~~True~~	~~True~~
False	False	True

Modus Tollens Examples

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.
I need to water my garden,
Therefore we’ve had less than 1 cm of rain in the past week.

Let p be the proposition, “We’ve had at least 1 cm of rain in the past week.”
Let q be the proposition, “I need to water my garden.”

Modus Tollens Examples

Let p be the proposition, “We’ve had at least 1 cm of rain in the past week.”
Let q be the proposition, “I need to water my garden.”

$p \rightarrow \sim q$
q
$\therefore \sim p$

Modus Tollens Common Mistakes

- Assuming ~p and drawing a conclusion.
- Assuming q and drawing a conclusion.

Modus Tollens Common Mistake

If p then q
Not p
Therefore Not q

All humans are mortal.
Socrates is not human,
Therefore Socrates is not Mortal.

Modus Tollens Common Mistake

If p then q
Not p
Therefore Not q

All humans are mortal.
Socrates is not human,
Therefore Socrates is not Mortal.

Socrates might be a cat, or an elephant, or a fish, or a mushroom, or a…
“False implies anything.”

Modus Tollens Common Mistake

Modus Tollens is a tautology.

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

“False implies anything.”

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

We haven’t had 1 cm of rain in the past week,

Therefore I need to water my garden.

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

We haven’t had 1 cm of rain in the past week,

Therefore I need to water my garden.

Modus Tollens Mistake Example

If p then q

q

$\therefore p$

All humans are mortal.

Socrates is mortal,

Therefore Socrates is a human.

Modus Tollens Mistake Example

If p then q

q

∴p

All humans are mortal.

Socrates is mortal,

Therefore Socrates is a human.

Socrates might be a turtle, a jellyfish, a fern...

Modus Tollens Common Mistake

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

I don’t need to water my garden,

Therefore we’ve had at least one cm of rain in the past week.

Modus Tollens Mistake Example

If we’ve had at least 1 cm of rain in the past week, I don’t need to water my garden.

I don’t need to water my garden,

Therefore we’ve had at least one cm of rain in the past week.

Perhaps I already watered my garden yesterday. Perhaps it is winter. Perhaps...

Disjunctive Syllogism

$p \vee q$

$\sim p$

$\therefore q$

Either Socrates is mortal or Socrates is not a human.
Socrates is not mortal,
Therefore Socrates is not a human.

Hypothetical Syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

All humans are animals.
All animals are mortal,
Therefore all humans are mortal.

Constructive Dilemma

$(p \to q)(r \to s)$
$p \vee r$
$\therefore q \vee s$

All dogs go to heaven and all cats have YouTube channels.
Abby is a dog or Dorothy is a cat,
Therefore Abby will go to heaven or Dorothy has a YouTube channel.

Absorption

$p \rightarrow q$
$\therefore p \rightarrow pq$

If Abby is a dog then Abby will go to heaven when she dies.

Therefore, if Abby is a dog, then Abby is a dog and Abby will go to heaven when she dies.

Simplification

pq
∴p

Abby is a dog and Abby will go to heaven when she dies.

Therefore Abby is a dog.

Conjunction

p

q

∴pq

Abby is a dog.

Abby will go to heaven when she dies.

Therefore Abby is a dog and Abby will go to heaven when she dies.

Addition

p
$\therefore p \lor q$

Abby is a dog.

Therefore Abby is a dog or humans live on Mars.

Example

Show the justification for the following argument.

$p \to q$

$r \to \sim q$

$\therefore p \to \sim q$

$p \to q$ $r \to \sim q$	

Example

Show the justification for the following argument.

$p \rightarrow q$
$r \rightarrow \sim q$
$\therefore p \rightarrow \sim r$

$p \rightarrow q$ $r \rightarrow \sim q$	
$\sim r \vee \sim q$	Material Implication
$\sim q \vee \sim r$	Commutativity
$q \rightarrow \sim r$	Material Implication
$(p \rightarrow q)(q \rightarrow \sim r)$	Conjunction
$\therefore p \rightarrow \sim r$	Hypothetical Syllogism

Problem

Prove the arguments

$p \rightarrow q$

$\sim(q \vee r)$

$\therefore \sim p$

Solution

Prove the validity of the argument

p→q
~(q∨r)
∴~p

p→q ~(q∨r)	
~q~r	DeM
~q	Simplification
~p	M.T.

Example - bringing it all together

Prove the validity of the argument.

If he signs the bill the governor will antagonize labor, and if he antagonizes labor he can’t be elected. But if he vetoes the bill he will break with the majority of his own party. The governor will not be elected if he breaks with the majority of his party. The governor must be reelected or his political career is over. He must either sign the bill or veto it. Therefore, the governor’s political career is finished.

Example - bringing it all together

Prove the validity of the argument.

If he signs the bill the governor will antagonize labor, and if he antagonizes labor he can't be elected. But if he vetoes the bill he will break with the majority of his own party. The governor will not be elected if he breaks with the majority of his party. The governor must be reelected or his political career is over. He must either sign the bill or veto it. Therefore, the governor's political career is finished.

S	He signs the bill
V	He vetoes the bill
L	He antagonizes labor
M	He breaks with the majority
R	He is reelected
O	His career is over

Example - bringing it all together

$S \rightarrow L$

$L \rightarrow \sim R$

$V \rightarrow M$

$M \rightarrow \sim R$

$\sim R \rightarrow O$

$S \vee V$

$\therefore O$

S	He signs the bill
V	He vetoes the bill
L	He antagonizes labor
M	He breaks with the majority
R	He is reelected
O	His career is over

Example - bringing it all together

1. S→L
2. L→~R
3. V→M
4. M→~R
5. ~R→O
6. S∨V

∴O

7. S→~R	Hypothetical Syllogism (1,2)
8. V→~R	Hypothetical Syllogism (3,4)
9. ~R∨~R	Constructive Dilemma (6,7,8)
10. ~R	Constant Evaluation (9)
∴O	Modus Ponens (5,9)

Problem

Prove the validity of the following argument.

$(p \vee q) \rightarrow pq$
$\therefore (p \rightarrow q)(q \rightarrow p)$

$(p \vee q) \rightarrow p$	Simplification
$(p \vee q) \rightarrow q$	Simplification
$\sim(p \vee q) \vee p$	M.I.
$\sim p \sim q \vee p$	De M.
$\sim q \vee p$	Truth Table
$q \rightarrow p$	M.I.
$p \rightarrow q$	Same reasoning as above
$(q \rightarrow p)(p \rightarrow q)$	Conjunction

Problem

Prove the validity of the following argument.

If Raphael joins the team then the team will have better fielding.
If Suhil joins the team then the team will have better batting.
Either Raphael or Suhil will join the team.
If the team has better fielding, then Suhil will join the team.
If the team has better batting, then Zhou will join the team.
Therefore either Suhil or Zhou will join the team.

Solution

- Let p be the proposition, “Raphael joins the team.”
- Let q be the proposition, “Suhil joins the team.”
- Let r be the proposition, “Zhou joins the team.”
- Let s be the proposition, “The team will have better fielding.”
- Let t be the proposition, “The team will have better batting.”

$p \rightarrow s$
$q \rightarrow t$
$p \vee q$
$s \rightarrow q$
$t \rightarrow r$
$\therefore q \vee r$

Solution

(1) p→s (2) q→t (3) p∨q (4) s→q (5) t→r ∴q∨r	
(6) (p→s)(s→q)	Conj. (1)(4)
(7) p→q	Hyp. Syl.(6)
(8) (q→t)(t→r)	Conj. (2)(5)
(9) q→r	Hyp. Syl. (8)

(10) (p→q)(q→r)(p∨q)	Conj. (3)(7)(9)
∴q∨r	Cons. Dil. (10)

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Validity

An argument is **valid** if there is no way for the premises to be true and the conclusion false.

An argument is **invalid** if the premise(s) can be true and the conclusion false.

Validity

An argument is **valid** if there is no way for the premises to be true and the conclusion false.

An argument is **invalid** if the premise(s) can be true and the conclusion false.

Only **valid** arguments can be used as logical proofs.

Invalid Argument Example

$p \rightarrow q$

$\sim p$

$\therefore \sim q$

All humans are mortal.

Socrates is not a human,

Therefore Socrates is not mortal

Invalid Argument Example

p	q	p→q
True	True	True
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

$p \rightarrow q$
$\sim p$
$\therefore \sim q$

1. Eliminate all rows that would make the major premise **false**.

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
False	False	True

$p \rightarrow q$

$\sim p$

$\therefore \sim q$

1. Eliminate all rows that would make the major premise **false**.
2. Eliminate all rows that would make the minor premise **false**.

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

$p \rightarrow q$
$\sim p$
$\therefore \sim q$

1. Eliminate all rows that would make the major premise **false**.
2. Eliminate all rows that would make the minor premise **false**.
3. Eliminate all rows that would make the conclusion **true**.

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

p → q
~p
∴ ~q

1. Eliminate all rows that would make the major premise **false**.
2. Eliminate all rows that would make the minor premise **false**.
3. Eliminate all rows that would make the conclusion **true**.

Any remaining rows show a case where the premises are true and the conclusion false.

Example

p	q	p→q
True	True	True
True	False	False
False	True	True
False	False	True

$p \rightarrow q$
q
$\therefore p$

Identify a case where the premises are true and the conclusion false.

Invalid Argument Example

p	q	p→q
~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~
False	True	True
~~False~~	~~False~~	~~True~~

$p \rightarrow q$
q
$\therefore p$

It is possible for the premises to be true and the conclusion false, so this is an **invalid** argument.

Example

Find a counter example demonstrating the argument is invalid.

All humans are mortal.
Socrates is mortal,
Therefore Socrates is a human.

Example

All humans are mortal.
Socrates is mortal,
Therefore Socrates is a human.

Counterexample: Socrates is a cat.
All humans are mortal. ✓
Socrates is mortal. ✓
Socrates is human. ✗

A **single counterexample** where the premises are true and the conclusion is false invalidates the argument.

Example

$p \rightarrow q$ $r \rightarrow s$ $p \vee s$ $\therefore q \vee r$	Find a set of truth value assignments where $q \vee r$ is false and the premises are all true.

Invalid Arguments Example

$p \rightarrow q$ $r \rightarrow s$ $p \vee s$ $\therefore q \vee r$	Find a set of truth value assignments where $q \vee r$ is false and the premises are all true.
$\sim(q \vee r)$	Assumption
$\sim q \sim r$	De Morgan’s
$\sim p$	Modus Tollens
s	Disjunctive Syllogism
p:False, q:False, r:False, s:True	
∴The argument is invalid	

Problem

Use a truth table to determine whether the following argument is valid or invalid.

$p \rightarrow q$

$q \rightarrow p$

$\therefore p \vee q$

Problem

Use a truth table to determine whether the following argument is valid or invalid.

$p \rightarrow q$
$q \rightarrow p$
$\therefore p \vee q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$
True	True	True	True	True
True	False	False	True	True
False	True	True	False	True
False	False	True	True	False

Solution

Use a truth table to determine whether the following argument is valid or invalid.

$p \rightarrow q$
$q \rightarrow p$
$\therefore p \lor q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \lor q$
True	True	True	True	True
~~True~~	~~False~~	~~False~~	~~True~~	~~True~~
~~False~~	~~True~~	~~True~~	~~False~~	~~True~~
False	False	True	True	False

Eliminate combinations where the premises are **false**.

Solution

Use a truth table to determine whether the following argument is valid or invalid.

$p \to q$
$q \to p$
$\therefore p \lor q$

p	q	$p \to q$	$q \to p$	$p \lor q$
~~True~~	~~True~~	~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~	~~True~~	~~True~~
~~False~~	~~True~~	~~True~~	~~False~~	~~True~~
False	False	True	True	False

Eliminate combinations where the conclusion is **true**.

Solution

Use a truth table to determine whether the following argument is valid or invalid.

$p \rightarrow q$
$q \rightarrow p$
$\therefore p \vee q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$
~~True~~	~~True~~	~~True~~	~~True~~	~~True~~
~~True~~	~~False~~	~~False~~	~~True~~	~~True~~
~~False~~	~~True~~	~~True~~	~~False~~	~~True~~
False	False	True	True	False

When p is false and q is false, the premises are both true and yet the conclusion is false. Therefore, this argument is invalid.

Problem

- If Deb is smart and works hard, then she will get a good review and will get promoted.
- If Deb is a bit dim-witted but works hard, she will be a valued employee.
- If Deb is a valued employee, she will get promoted.
- If Deb is smart, she will work hard.
- Therefore, Deb will get promoted.

Solution

- If Deb is smart and works hard, then she will get a good review and will get promoted.
- If Deb is a bit dim-witted but works hard, she will be a valued employee.
- If Deb is a valued employee, she will get promoted.
- If Deb is smart, she will work hard.
- Therefore, Deb will get promoted.

Let S represent Deb is smart.
Let W represent Deb works hard.
Let R represent Deb gets a good review.
Let P represent Deb gets promoted.
Let V represent Deb is a valued employee.

$SW \rightarrow RP$
$\sim SW \rightarrow V$
$V \rightarrow P$
$S \rightarrow W$
$\therefore P$

Solution

(1) SW→RP (2) ~SW→V (3) V→P (4) S→W ∴P	
(5) ~P	Assumption
(6) ~V	M.T. (3)
(7) ~(RP)	C.E. (5)
(8) ~(SW)	MT (1)(7)
(9) ~S ∨ ~W	DeM (8)

(10) S →~W	M.I. (9)
(11) ~S	T.T. (4)(9)
(12) ~(~SW)	MT (2)(6)
(13) S ∨ ~W	DeM
(14) W	CE (11)(13)

S: False, V: False, P: False, W: True, R: N/A

Valid Arguments

All unicorns have white fur.
Lorenzo is a unicorn,
Therefore Lorenzo has white fur.

It doesn’t matter if there are no unicorns. The argument is valid because there is no way for the premises to be true and the conclusion false.

Soundness

An argument is sound if it is valid and the premises are true.

Unsound Arguments

All humans are fish.
Socrates is a human,
Therefore Socrates is a fish.

Unsound Arguments

All humans are fish.
Socrates is a human,
Therefore Socrates is a fish.

Valid!

Unsound Arguments

All humans are fish. ✘
Socrates is a human,
Therefore Socrates is a fish.

Valid!

Unsound.

Unsound Arguments

All unicorns have white fur.
Lorenzo is a Unicorn,
Therefore Lorenzo has white fur.

Unsound Arguments

All unicorns have white fur.
Lorenzo is a Unicorn,
Therefore Lorenzo has white fur.

Valid!

Unsound Arguments

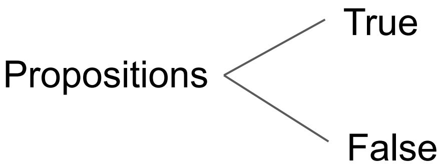
All unicorns have white fur.
Lorenzo is a Unicorn, ✘
Therefore Lorenzo has white fur.

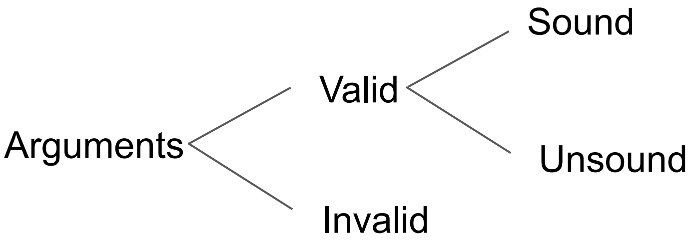
Valid!

Unsound.

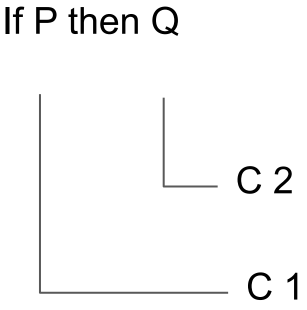
Symbolic (Formal) Logic Addresses the *Validity* of Arguments, not the *Soundness* of Arguments

Terminology





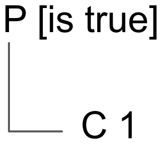
Major Premise : defines a relationship between two distinct argument components.



Relationship:

If...then
(Implication)

Minor Premise: provides information about one of the components from the major premise.



Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Proof by Contradiction

Assume the opposite of what you want to prove.
Show a contradiction.

$p \lor q \to r$
p
$\therefore r$

Use proof by contradiction.

Proof by Contradiction

p∨q→r p ∴r	
~r	Assumption
~(p∨q)	Modus Tollens
~p~q	De Morgan’s
~p	Simplification
p~p	Conjunction
∴r	Proof by Contradiction

Example

Use proof by contradiction to prove the conclusion.

$(p \vee q) \rightarrow rst$

q

$\therefore rs$

Example

$(p \lor q) \to (rs)t$ q $\therefore rs$	
$\sim(rs)$	Assumption
$\sim[(rs)t]$	Constant Evaluation
$\therefore \sim(p \lor q)$	M.T.
$\sim p \sim q$	De Morgan’s
$\sim q$	Simplification
$q \sim q$	Conjunction
$\therefore rs$	Proof by Contradiction

Problem

Use Proof by Contradiction to prove the validity of the following argument.

(1) pq

(2) $(p \vee r) \rightarrow s$

$\therefore ps$

Solution

(1) pq (2) (p ∨ r) → s ∴ps	
(3) ~(ps)	Assumption
(4) ~p ∨ ~s	DeM (3)
(5) ~s ∨ ~p	Comm. (4)
(6) s → ~p	M.I. (5)
(7) (p ∨ r) → ~p	H.S. (2)(6)
(8) ~p	TT (7)
(9) ~(pq)	CE (8)
(pq)~(pq)	Conj (1)(9)
∴ps	Proof by Contradiction

Problem

Use Proof by Contradiction to demonstrate the validity of the following argument.

(1) $(p \lor q) \rightarrow rs$

(2) $(r \lor s) \rightarrow t$

(3) p

$\therefore t$

Solution

(1) $(p \lor q) \rightarrow rs$ (2) $(r \lor s) \rightarrow t$ (3) p $\therefore t$	
(4) ~t	Assumption
(5) ~(r ∨ s)	M.T. (2)(4)
(6) ~r~s	De.M. (5)
(7) ~r	Simp (6)
(8) ~(rs)	C.E. (7)

(9) ~(p ∨ q)	M.T. (1)(8)
(10) ~p~q	DeM (9)
(11) ~p	Simp (10)
(12) p~p	(3)(11)
∴ t	Proof by Contradiction

Problem

Use proof by contradiction to prove the validity of the following argument.

If Brown received the email then she took a flight. If she took a flight, then she will attend the conference. If the email bounced, then she will miss the conference. Either Brown received the email, or the email bounced. Therefore, either Brown caught the flight, or she missed the conference.

Solution

Let E represent Brown received the email
Let F represent Brown took a flight
Let B represent the email bounced
Let C represent Brown attends the conference

$E \rightarrow F$

$F \rightarrow C$

$B \rightarrow \sim c$

$E \lor B$

$\therefore F \lor \sim C$

Solution

(1) E → F (2) F→ C (3) B → ~C (4) E ⋁ B ∴ F ⋁ ~C	
(5) ~(F ⋁ ~C)	Assumption
(6) ~FC	De.M. (5)
(7) ~F	Simp. (6)
(8) ~E	M.T. (1)(7)
(9) C	Simp (6)

(10) ~B	M.T. (3)(9)
(11) ~E~B	Conj. (8)(10)
(12) ~(E ⋁ B)	De.M. (11)
(13) ~(E ⋁ B)(E ⋁ B)	Conj. (4)(12)
∴ F ⋁ ~C	Proof by Contradiction

Proof Techniques

- List all possible values. (Truth Tables)
- Deduce using rules of inference.
- Demonstrate invalidity.
- Proof by contradiction.
- Prove equivalence, tautology, contradiction by converting into a normal form.

Normal Forms

- Sum of products
- Product of sums

Sum of Products

All propositions are in the form $p_1 \vee p_2 \vee p_3 \vee .... \vee p_n$ Where $p_1 ... p_n$ are in the form $p_1' \wedge p_2' \wedge p_3' \wedge ... p_n'$

Example: Transform $(p \rightarrow q) \wedge r$ into sum of products form.

$(p \rightarrow q)r$	
$(\sim p \vee q)r$	Material Implication
$\sim pr \vee qr$	Distribution

Product of Sums

All propositions are in the form $p_1 \wedge p_2 \wedge p_3 \wedge ... p_n$ Where $p_1 ... p_n$ are in the form $p'_1 \vee p'_2 \vee p'_3 \vee .... \vee p'_n$

Example: Transform (p→q)(q→r) into product of sums form.

(p→q)(q→r)	
(~p∨q)(q→r)	Material Implication
(~p∨q)(~q∨r)	Material Implication

Canonical Forms

A normal form is a canonical form if each term contains **all symbols** being used.

- Two propositions with the same canonical forms are equal.
- Given a logic expression with n distinct symbols in canonical form,

Number of Terms with n symbols	Sum of Products	Product of Sums
0	Contradiction	Tautology
2^n	Tautology	Contradiction
$0 < \text{n-terms} < 2^n$	Contingent	Contingent

Example

Transform (p→q)r into sum of products canonical form.

(p→q)r	
(~p∨q)r	M.I.
~pr∨qr	Dist.
~pr(q∨~q) ∨ qr(p∨~p)	CE
~pqr ∨ ~p~qr ∨ pqr ∨ ~pqr	Dist
~pqr ∨ ~p~qr ∨ pqr	Identity
pqr ∨ ~pqr ∨ ~p~qr	Comm.

Example

Show ~(p∨r)→~(qr) is a tautology using sum of products canonical form.

~(p∨r)→~(qr)	
~~(p∨r)∨~(qr)	Material Implication
p∨~q∨r∨~r	Double Negation/De Morgan's
p(q∨~q)(r∨~r)∨~q(p∨~p)(r∨~r)∨r(p∨~p)(q∨~q)∨~r(p∨~p)(q∨~q)	Excluded Middle
pqr∨pq~r∨p~qr∨p~q~r∨~pqr∨~pq~r∨~p~qr∨~p~q~r	Distribution/Identity

Example

Show (p→q)∧p∧~q is a contradiction using sum of products canonical form.

(p→q)p~q	
(~p∨q)p~q	Material Implication
~pp~q∨pq~q	Distribution
null	Non-contradiction/Evaluation

Problems

Convert the following to SOP Normal Form and SOP Canonical Form.

1. $p \lor (\sim pq)$
2. $q(p \lor \sim q)$

Use SOP Canonical Form to determine whether the propositions are equivalent.

1. $(p \lor q) \rightarrow [(p \rightarrow q) \rightarrow \sim(q \lor \sim q)]$
2. $(p \rightarrow q) \rightarrow \sim(p \lor q)$

Solutions

p ∨ (~pq)	
p(q∨~q)∨ (~pq)	C.E.
pq ∨(p~q)∨ (~pq)	Dist.

q(p ∨ ~q)	qp∨~qq
qp∨~qq	Dist.
qp∨False	Non. Cont.
qp	Const. Eval.

Solutions

(p⋁q)→ [(p→q)→~(q ⋁ ~q)]	
~(p⋁q)⋁ [(p→q)→~(q ⋁ ~q)]	M.I.
~(p⋁q)⋁~(p→q)⋁ ~(q ⋁ ~q)	M.I.
~(p⋁q)⋁~(p→q)	E.M./C.E.
~(p⋁q)⋁~(~p⋁q)	M.I.
~p~q⋁p~q	De.M.

(p→q)→~(p⋁q)	
(p→q)→~p~q	De.M.
~(p→q)⋁~p~q	M.I.
~(~p⋁q)⋁~p~q	M.I.
p~q⋁~p~q	De.M.
~p~q⋁p~q	Comm.

Prepositional Logic

Prepositional Logic

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

Prepositional Logic

All humans are mortal.
Socrates is a human,
Therefore Socrates is mortal.

We cheated!

All humans are mortal,
Therefore, if Socrates is a human, then Socrates is Mortal.
Socrates is a human,
Therefore Socrates is mortal.

Prepositional Logic

Jargon Alert

Attributes/Predicates

All Humans are Mortal.
Socrates is a Human,
Therefore Socrates is Mortal.

Human is an attribute (predicate) of some x.
Hx

Mortal is an attribute (predicate) of some x.
Mx

Attributes/Predicates

Human is a predicate of some x.
Hx

Socrates is Human.
Hs

Mortal is a predicate of some x.
Mx

Socrates is Mortal.
Ms

Attributes/Predicates

H denotes the predicate Human.

Hs

Instance s denotes some specific thing to which H applies. E.g. Socrates is human.

Φ denotes some unspecified predicate

Φx

Variable x denotes some unspecified instance to which predicate Φ applies.

Conventions.

- Upper-case A-Z letters denote predicates.
- Lower-case a-w denote specific thingsΦ to which the predicate applies.
- Lower-case x,y,z denote variables.
- Upper-case greek letters, e.g. ϕ, denote non-specific predicates.

Prepositions and Propositions

A proposition is a statement to which a truth value can be assigned.

Algebra	Prepositional Logic
Is 3 > x true or false?	Is Hx true or false?
3 > x is not a proposition.	Hx is not a proposition.
Instantiate with x = 5.	Instantiate with x = s where s represents the classical-era philosopher Socrates.
3 > 5 is false.	Hs is true.
3 > 5 is a proposition.	Hs is a proposition.

Prepositions and Propositions

Assigning a **specific value** to a preposition is called ***instantiation***.

Prepositions and Propositions

Assigning a **specific value** to a preposition is called ***instantiation***.

Instantiated predicates are propositions.

Hs: Socrates is a human.

Hx: x is a human.

Prepositions and Propositions

Assigning a **specific value** to a preposition is called ***instantiation***.

Instantiated predicates are propositions.

Hs: Socrates is a human. ✔

Hx: x is a human? ✘

Prepositions and Propositions

Two ways to convert a preposition into a proposition:

- Instantiation
- Quantification
  - Universal quantification $\forall$
  - Existential quantification $\exists$

Universal Quantification

$\forall(x)Hx$

Everything in the Universe is human.

For all x, x is human.

This is a proposition because we can assign a truth-value to the statement.

Existential Quantification

$\exists(x)Hx$

There is at least one thing in the universe that is human.

Something is human.

There exists an x such that x is human.

This is a proposition because we can assign a truth value to this statement.

Prepositions and Propositions

Statement	Proposition?	Truth-Value
Hx	✗	?
Hs	✓	True
∀(x)Hx	✓	False
∃(x)Hx	✓	True

Closure

Logic functions can be applied to prepositions. The results will also be prepositions.

Let Φx, Ψx be prepositions

- $\sim\Phi x$ is a preposition.
- $\Phi x \lor \Psi x$ is a preposition.
- $\Phi x \land \Psi x$ is a preposition.
- $\Phi x \rightarrow \Psi x$ is a preposition.

Quantifiers and Propositions

Quantifiers can be applied to prepositions.

The result is a proposition.

$\exists(x){\sim}\phi x$	There exists an x such that ${\sim}\phi x$
$\exists(x)[\phi x \vee \Psi x]$	There exists an x such that $\phi x \vee \Psi x$
$\exists(x)[\phi x \wedge \Psi x]$	There exists an x such that $\phi x \wedge \Psi x$
$\exists(x)[\phi x \rightarrow \Psi x]$	There exists an x such that $\phi x \rightarrow \Psi x$

Quantifiers and Propositions

Quantifiers can be applied to propositions.

The result is a proposition.

$\forall(x)\sim\phi x$	For all x, $\sim\phi x$
$\forall(x)[\phi x \vee \Psi x]$	For all x, $\phi x \vee \Psi x$
$\forall(x)[\phi x \wedge \Psi x]$	For all x, $\phi x \wedge \Psi x$
$\forall(x)[\phi x \rightarrow \Psi x]$	For all x, $\phi x \rightarrow \Psi x$

Examples

Convert the following sentences into propositions.

Cats speak French.	
Cats do not speak French.	
Siamese cats are haughty.	
Some cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	
Siamese cats are haughty.	
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C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	
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My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	$\exists(x)[Cx \wedge Lx]$
My cat Dorothy does not have fleas.	

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Convert the following sentences into propositions.

Cats speak French.	$\forall(x)[Cx \rightarrow Fx]$
Cats do not speak French.	$\forall(x)[Cx \rightarrow \sim Fx]$
Siamese cats are haughty.	$\forall(x)[Sx \rightarrow Hx]$
Some cats have fleas.	$\exists(x)[Cx \land Lx]$
My cat Dorothy does not have fleas.	$Cd \land \sim Ld$

C:is a cat. F:speaks French. S:is a Siamese cat. H: is haughty. L: Has fleas.

Examples

Existential quantifiers do not distinguish between probabilities.

Exactly one cat has fleas.
Most cats have fleas.
A few cats have fleas.
All the cats in the world except one have fleas.

Examples

Existential quantifiers do not distinguish between probabilities.

Exactly one cat has fleas.
Most cats have fleas.
A few cats have fleas.
All the cats in the world except one have fleas.

$\exists (x)[Cx \land Lx]$

Prepositions with Multiple Arguments

Nancy is taller than Marcia.

Let Tx,y mean x is taller than y.
Let n represent Nancy.
Let m represent Marcy.

Tn,m

Nested Quantified Prepositions

"Man is the measure of all things."

Let Hx mean x is human.
Let Ty mean y is a thing.
Let Mx,y mean x measures y.

$\forall(x)\forall(y)HxTy \rightarrow Mx,y$

Nested Quantified Prepositions

“Everybody loves somebody sometime.”

Let x,y,z be anything.
Let Hx mean x is a person and let Hy mean y is a person.
Let Tz be a specific point in time z.
Let Lx,y,z mean x loves y at time t.

$\forall(x)\exists(y)\exists(z)HxHyTzLx,y,z$

Nested Quantified Prepositions

Given any integer, there is always a larger integer and always a smaller integer.

Let x be any integer.
Let y be an integer.

$$\forall(x)\big[[\exists(y)x<y]\wedge\exists(y)[x>y]\big]$$

Nested Quantified Prepositions

Given any integer, there is always a larger integer and always a smaller integer.

Let x be any integer.
Let y be an integer.

$$\forall(x)\big[[\exists(y)x<y]\wedge\exists(y)[x>y]\big]$$

Variable x is bound to both instances.

Nested Quantified Prepositions

Given any integer, there is always a larger integer and always a smaller integer.

Let x be any integer.
Let y be an integer.

$\forall(x)\big[[\exists(y)x<y]\wedge\exists(y)[x>y]\big]$

Variable y is bound to a single instance.

Nested Quantified Prepositions

Given any integer, there is always a larger integer and always a smaller integer.

Let x be any integer.
Let y be an integer.

$$\forall(x)\big[[\exists(y)x<y]\land\exists(y)[x>y]\big]$$

Variable y is bound to a single instance.

Problems

Convert the following language statements into prepositional propositions.

1. You can’t always get what you want.
2. Whether you’re a brother or whether you’re a mother, you’re staying alive.
3. Girls just want to have fun.
4. I like big butts and I cannot lie.
5. I don't care for fancy things or to take part in a precious race.
6. I got the sunshine in my pocket.
7. I left my scarf there, at your sister’s house.

Solutions

You can't always get what you want.

Let x be a thing.
Let W_x be a thing you want.
Let G_x be a thing you get.

$\exists (x)W_x{\sim}G_x$

There is a thing x such that you want x but you do not get x.

Problems

Whether you're a brother or whether you're a mother, you're staying alive.

Let x be some entity.
Let B_x represent x is a brother.
Let M_x represent x is a mother.
Let A_x represent x is staying alive.

$\forall(x)(B_x \vee M_x) \rightarrow A_x$

For all entities, if the entity is either a brother or a mother, then the entity is staying alive.

Problems

Girls just want to have fun.

Let x be something.
Let F_x represent something that is fun.
Let W_x represent something a girl wants.

$\forall(x) \sim F_x \rightarrow \sim W_x$

If something is not fun, then a girl does not want it.

Problems

I like big butts and I cannot lie.

Let x be any utterance.
Let bb be the utterance, "I like big butts."
Let S_x be an utterance I say.
Let T_x be an utterance that is true

$\forall(x)S_x \rightarrow T_x$
$S_{bb} \rightarrow T_{bb}$
S_{bb}
$\therefore T_{bb}$

For all possible utterances, if I say the utterance, then that utterance is true. So if I say I like big butts, then it is true that I like big butts. I do say I like big butts, therefore it is true that I like big butts.

Problems

I don't care for fancy things or to take part in a precious race.

Let t be a thing.
Let r be a race.
Let Cx be something I care about.
Let Fy be a fancy thing.
Let Pz be a precious thing.
$\forall(t,r)(Ft \rightarrow {\sim}Ct)(Pr \rightarrow {\sim}Cr)$

For all things, if it is a fancy thing then I don’t care about it, and for all races, if it is a precious race, then I don’t care about it.

Problems

I got the sunshine in my pocket.

Let s be sunshine.
Let P_x be something in my pocket.
P_s

There is sunshine in my pocket.

Problems

I left my scarf there, at your sister's house.

Let s be a scarf.
Let h be your sister's house.
Let Mx be something that is mine.
Let Lx,y be something x left at place y.

MsLsh

The scarf is mine and I left the scarf at your sister's house.

Quantifiers and Negation

We have different rules for using negation in conjunction with quantifiers.

Let Mx represent the predicate, “x is mortal.”

Mx	x is mortal.
~Mx	x is not mortal.
∀(x)Mx	Everything is mortal.
~∀(x)Mx	Not everything is mortal.
∀(x)~Mx	Everything is not mortal.

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∀(x)Mx	Everything is mortal.
~∀(x)Mx	Not everything is mortal. At least one thing is not mortal.
∀(x)~Mx	Everything is not mortal. Nothing is mortal.

Quantifiers and Negation

We have different rules for using negation in conjunction with quantifiers.

Let Mx represent the predicate, “x is mortal.”

Mx	x is mortal.
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∃(x)Mx	At least one thing is mortal.
~∃(x)Mx	Nothing exists that is mortal.
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Quantifiers and Negation

We have different rules for using negation in conjunction with quantifiers.
Let Mx represent the predicate, “x is mortal.”

Mx	x is mortal.
~Mx	x is not mortal.
∃(x)Mx	At least one thing is mortal.
~∃(x)Mx	Nothing exists that is mortal. Everything is not mortal.
∃(x)~Mx	At least one thing is not mortal. It is not the case that everything is mortal.

Contradictories

Quantified Proposition	Contradictory
$\forall (x)\phi x$	$\sim\forall (x)\phi x$
$\exists (x)\phi x$	$\sim\exists (x)\phi x$

Contradictories

Quantified Proposition	Contradictory
$\forall(x)\phi x$	$\sim\forall(x)\phi x$ $\exists(x)\sim\phi x$
$\exists(x)\phi x$	$\sim\exists(x)\phi x$ $\forall(x)\sim\phi x$

Contradictories

Quantified Proposition	Contradictory
∀(x)ϕx	~~~∀(x)ϕx~~ ∃(x)~ϕx
∃(x)ϕx	~~~∃(x)ϕx~~ ∀(x)~ϕx

Contradictories

Quantified Proposition	
∀(x)Mx	Everything is Mortal
∃(x)Mx	At least one thing is mortal.

Contradictories

Quantified Proposition		Contradictory	
∀(x)Mx	Everything is Mortal	~∀(x)Mx ∃(x)~Mx	Not everything is mortal. At least one thing is not mortal.
∃(x)Mx	At least one thing is mortal.	~∃(x)ϕx ∀(x)~ϕx	It is not the case that at least one thing is mortal. Everything is not mortal.

Contradictories

Quantified Proposition		Contradictory	
∀(x)Mx	Everything is Mortal	~∀(x)Mx ∃(x)~Mx	At least one thing is not mortal.
∃(x)Mx	At least one thing is mortal.	~∃(x)Mx ∀(x)~Mx	Everything is not mortal.

Contradictory Examples

Using contradictories, modify each proposition to start with a quantifier rather than a negation.

~∀(x)[Ax→Bx]	
~∀(x)[~Ax~BX]	
~∃(x)[Ax→~Bx]	

Contradictory Examples

Using contradictories, modify each proposition to start with a quantifier rather than a negation.

~∀(x)[Ax→Bx]	∃(x)~[Ax→Bx] ∃(x)Ax~Bx
~∀(x)[~Ax~BX]	∃(x)~[~Ax~Bx] ∃(x)[Ax∨Bx]
~∃(x)[Ax→~Bx]	∀(x)~[Ax→~Bx] ∀(x)AxBx

Contraries

$\forall(x)\phi x$ and $\forall(x){\sim}\phi x$ are contraries.

Example:
Let Hx represent the predicate, “x is human.”

$\forall(x)Hx$ is false because not everything is human.
$\forall(x){\sim}Hx$ is false because some things are human.

Contraries can both be false, but can’t both be true.

Subcontraries

$\exists(x)\phi x$ and $\exists(x)\sim\phi x$ are subcontraries.

Example:
Let Hx represent the predicate, “The thing x is human.”

$\exists(x)Hx$ is true because at least one thing is human.
$\exists(x)\sim Hx$ is true because at least one thing is not human.

Subcontraries can both be true, but can’t both be false.

Examples

Construct the contrary and contradictory propositions.

Proposition	All politicians are honest.	∀(x)[Px→Hx]
Contrary	All politicians are not honest.	∀(x)~[Px→Hx] ∀(x)Px~Hx
Contradictory	Not all politicians are honest. Some politicians are dishonest. There is at least one dishonest politician.	∃(x)[Px~Hx]

Examples

Construct the subcontrary and contradictory propositions.

Proposition	Some politicians are honest.	$\exists (x)[PxHx]$
Subcontrary	Some politicians are not honest.	$\exists (x)\sim[PxHx]$
Contradictory	It is not the case that some politicians are honest. All politicians are dishonest.	$\forall (x)[Px \rightarrow \sim Hx]$

Example

Translate the statement into a quantified predicate. Define its contrary and its contradictory, and then translate the results back into natural language.

All cats are arrogant.

Example

All cats are arrogant.

Let Cx represent, ‘x is a cat.’
Let Ax represent, ‘x is arrogant.’

∀(x)[Cx→Ax]	∀(x)~[Cx→Ax] ∀(x)[Cx~Ax]	~∀(x)[Cx→Ax] ∃(x)[Cx~Ax]
For all things x, if x is a cat then x is arrogant.	Nothing is both a cat and arrogant.	There is at least one thing that is both a cat and not arrogant.

Example

Translate the statement into a quantified predicate. Define its subcontrary and its contradictory, and then translate the results back into natural language.

Some cats are arrogant.

Example

Some cats are arrogant.

Let Cx represent, ‘x is a cat.’
Let Ax represent, ‘x is arrogant.’

∃(x)CxAx	∃(x)~[CxAx]	~∃(x)[CxAx] ∀(x)~[CxAx] ∀(x)[Cx→~Ax]
At least one thing that is a cat is also arrogant.	There is at least one thing that is not both a cat and arrogant.	All cats are not arrogant.

Problems

Convert the language to a quantified proposition. Define its contradictory and either contrary or subcontrary as appropriate. Then translate back to language.

1. M&Ms melt in your mouth, not in your hands.
2. Choosy mothers choose Jiff.
3. So easy a caveman can do it.

Solutions

M&Ms melt in your mouth, not in your hands.

Solution 1:

Let x be anything
Let Cx be something that is an M&M candy
Let Mx be something that melts in your mouth
Let Hx be something that melts in your hand

$\forall(x)\ Cx \rightarrow Mx{\sim}Hx$

Solution 2:

Let x be an M&M candy
Let Mx be an M&M that melts in your mouth
Let Hx be an M&M that melts in your hand

$\forall(x)\ Mx{\sim}Hx$

Solution

$\forall(x)\ Cx \rightarrow Mx{\sim}Hx$

Contradictory:

$\sim\forall(x)\ Cx \rightarrow Mx{\sim}Hx$

$\exists(x)\ {\sim}(Cx \rightarrow Mx{\sim}Hx)$

$\exists(x)Cx{\sim}Mx \vee CxHx$

Contrary:

$\forall(x)\ {\sim}(Cx \rightarrow Mx{\sim}Hx)$

$\forall(x)Cx{\sim}Mx \vee CxHx$

$\forall(x)\ Mx{\sim}Hx$

Contradictory:

$\sim\forall(x)Mx{\sim}Hx$

$\exists(x)\ {\sim}Mx \vee Hx$

Contrary:

$\forall(x){\sim}Mx \vee Hx$

Solution

$\forall(x)\ Cx \rightarrow Mx{\sim}Hx$

For all x, if x is an M&M then x melts in your mouth and x does not melt in your hand.

$\exists(x)Cx{\sim}Mx \vee CxHx$

There is at least one x that is an M&M that does not melt in your mouth or does melt in your hand.

$\forall(x)Cx{\sim}Mx \vee CxHx$

For all x, either x is an M&M and does not melt in your mouth, or x is an M&M and does melt in your hand.

$\forall(x)\ Mx{\sim}Hx$

For all M&Ms, the M&M does melt in your mouth and does not melt in your hand.

$\exists(x)\ {\sim}Mx \vee Hx$

There is some M&M that does not melt in your mouth or does melt in your hand.

$\forall(x){\sim}Mx \vee Hx$

For all M&Ms, either it does not melt in your mouth or it does melt in your hand.

Solutions

Choosy mothers choose Jiff.

Let x be a human mother
Let Mx be a choosy human mother
Let Cx be Jiff peanut butter being chosen by x

$\forall(x)Mx \rightarrow Cx$

Solutions

Quantified Predicate	Contradictory	Contrary
∀(x)Mx → Cx	∃(x) Mx~Cx	∀(x)~(Mx → Cx)
For all mothers, if the mother is choosy, then the mother chooses Jiff.	There is at least one mother who is both choosy and does not choose Jiff.	For all mothers, it is not the case that if the mother is choosy, then the mother chooses Jiff.

Solutions

So easy a caveman can do it.

Let x represent a task
Ex represents a task that is easy enough for a caveman to perform.

$\exists (x)\ Ex$

Solutions

Quantified Predicate	Contradictory	Subcontrary
$\exists (x)\ Ex$	$\forall (x) {\sim} Ex$	$\exists (x)\ {\sim} Ex$
There is at least one task that is easy enough for a caveman to perform.	For all possible tasks, there is no such task that a caveman can perform.	There is at least one task too difficult for a caveman to perform.

Reasoning with Quantifiers

All humans are mortal.
If Socrates is a human, then Socrates is mortal.
Socrates is human,
Therefore, Socrates is mortal.

How do we get from here...

To here?

Reasoning with Quantifiers

All roads lead to Rome.
Burns Lane is a road,
Therefore Burns Lane leads to Rome.

Reasoning with Quantifiers

All roads lead to Rome.
Burns Lane is a road,
Therefore Burns Lane leads to Rome.

$\forall(x)Rx \rightarrow Lx$
Rb

Reasoning with Quantifiers

All roads lead to Rome.
Burns Lane is a road,
Therefore all highways lead to Rome.

$\forall(x)Rx \rightarrow Lx$
Rb

$\forall(x)Rx \rightarrow Lx$ Rb	Can we just drop the quantifier?
$Rb \rightarrow Lb$ Rb	
$\therefore Lb$	M.P.

Reasoning with Quantifiers

All roads lead to Rome.
Burns Lane is a road,
Therefore all highways lead to Rome.

$\forall(x)Rx \rightarrow Lx$
Rb

$\forall(x)Rx \rightarrow Lx$ Rb	Can we just drop the quantifier?
$Rb \rightarrow Lb$ Rb	
$\therefore Lb$	M.P.

This looks like it works, however...

Reasoning with Quantifiers

Some people have white hair.
Some people do not have white hair.
If a person has white hair, the person does not have brown hair.
Shaw has white hair.

Reasoning with Quantifiers

Some people have white hair.
Some people do not have white hair.
If a person has white hair, the person does not have brown hair.
Shaw has white hair.

∃(x)~Wx ∃(x)Wx ∀(x)Wx→~Bx Ws	
~Ws Ws Ws→~Bs	Can we just drop the quantifiers?
Ws~Ws	Conjunction

Reasoning with Quantifiers

Some people have white hair.
Some people do not have white hair.
If a person has white hair, the person does not have brown hair.
Shaw has white hair.

∃(x)~Wx ∃(x)Wx ∀(x)Wx→~Bx Ws	
~Ws Ws Ws→~Bs	Can we just drop the quantifiers?
Ws~Ws	Conjunction

Quantifier Rules of Inference

- Universal Instantiation
- Universal Quantification
- Existential Instantiation
- Existential Quantification

Universal Instantiation

Any universally quantified proposition must be true for all instances of the quantified variable.

Rainy Days and Mondays always get me down.
Today is rainy,
Therefore today I am down.

$\forall(x)[(Rx \lor Mx) \rightarrow Ds]$

If x is a rainy day or x is a Monday, then Shaw is down.

Universal Instantiation

$\forall(x)[(Rx \lor Mx) \rightarrow Ds]$

Let d be a day and assume it is rainy. Instantiate the universally quantified proposition with the instance d.

$(Rd \lor Md) \rightarrow Ds$
Rd
$\therefore Ds$

Because $(Rx \lor Mx) \rightarrow Ds$ is true for **all** instances of x, it must be true for the instance of x that is the day represented by the symbol d.

Universal Instantiation: Substitute an ***instance*** of x for the quantified x.

Example

Assume all roads lead to Rome.
Assume Burns Lane is a road.

∀(x)[Rx→Lx] Rb	
Rb→Lb	Universal Instantiation
∴Lb	Modus Ponens

Therefore, Burns Lane leads to Rome.

Contradictories Revisited

Let ∀(x)Hx represent the proposition, "Everything is Human."

Let ~∀(x)Hx represent the proposition, "Not everything is Human."

Use Universal Instantiation. Let x=s where s is Kelly Shaw, resident of Williamsburg, VA.

Negation does not work correctly when applied to a universally quantified predicate. Use its contradictory conversion instead.

~[∀(x)Hx] Hs	Not everyone is human. Shaw is human.
~[Hs]	U.I.
∴~HsHs	Shaw is both human and not human?

Universal Generalization

If a proposition is true for all possible instances, then it can be universally generalized and quantified.

1. Instantiate universal quantifiers with a *general* rather than a *specific* instance.
2. Deduce some other preposition from the result.
3. Generalize to all instances using **Universal Generalization**.

ϕy	Where y is **any** possible instance.
$\therefore \forall(x)\phi x$	Where x is **every** possible instance.

Universal Generalization

Example:

Assume all roads lead to Rome.
Assume all highways are roads.
Prove all highways lead to Rome.

$\forall(x)[Rx \rightarrow Lx]$
$\forall(x)[Hx \rightarrow Rx]$

Prove: $\forall(x)[Hx \rightarrow Lx]$

Universal Generalization

$\forall(x)[Rx \rightarrow Lx]$
$\forall(x)[Hx \rightarrow Rx]$

Instantiate with a *general* rather than a *specific* variable.

Let y be **any** Highway.

Universal Generalization

$\forall(x)[Rx \rightarrow Lx]$ $\forall(x)[Hx \rightarrow Rx]$ Hy	Assume y is any highway
$Hy \rightarrow Ry$	Universal Instantiation
$\therefore Ry$	Modus Ponens
$Ry \rightarrow Ly$	Universal Instantiation
$\therefore Hy \rightarrow Ly$	Hypothetical Syllogism
$\therefore \forall(x)Hx \rightarrow Lx$	Universal Generalization

Existential Instantiation

An existential quantifier implies an instance.

$\exists(x)\phi x$

$\therefore \phi z$

Be sure z is not used in any other context within the proof.

Existential Generalization

If a preposition is true, then it can be existentially quantified.

ϕz

$\therefore \exists (x)\phi x$

Existential Instantiation/Generalization

Example:

All humans are mortal.
Humans exist,
Therefore mortal beings exist.

$\forall(x)[Hx \rightarrow Mx]$
$\exists(x)Hx$
Prove: $\exists(x)Mx$

Existential Instantiation/Generalization

$\forall(x)[Hx \rightarrow Mx]$ $\exists(x)Hx$	Assume a is a human
Ha	Existential Instantiation
$Ha \rightarrow Ma$	Universal Instantiation
$\therefore Ma$	Modus Ponens
$\therefore \exists(x)Mx$	Existential Generalization

Existential Instantiation Common Mistakes

Existentially quantified premises instanted with general instances must be done in multiple steps to insure **all instances are different**.

Example:

Some people eat meat
Some lions eat meat
Therefore some people are lions?

Existential Instantiation Common Mistakes

∃ (x)PxMx ∃ (x)LxMx	
PaMa LaMa	Existential Instantiation
Pa	Simplification
La	Simplification
PaLa	Conjunction
∴ ∃ (x)PxLx	Existential Generalization

Existential Instantiation Common Mistakes

∃ (x)PxMx ∃ (x)LxMx	
PaMa LaMa	~~Existential Instantiation~~
Pa	Simplification
La	Simplification
PaLa	Conjunction
∴ ∃ (x)PxLx	Existential Generalization

Existential Instantiation Common Mistakes

∃(x)PxMx ∃(x)LxMx	
PaMa	Existential Instantiation
LbMb	Existential Instantiation

Existential Quantification Contradictories Revisited

No humans are immortal.
Therefore all humans are mortal.

Let Hx represent x is human.
Let Mx represent x is mortal.

$\sim\exists(x)Hx{\sim}Mx$
$\therefore\forall(x)Hx\rightarrow Mx$

Existential Quantification Contradictories Revisited

No humans are immortal.
Therefore all humans are mortal.

Let Hx represent x is human.
Let Mx represent x is mortal.

~∃(x)Hx~Mx
∴∀(x)Hx→Mx

~[∃(x)Hx~Mx]	
~[(Ha~Ma)]	E.I.
~Ha∨Ma	De M.
Ha→Ma	M.I.
∃(x)Hx→Mx	E.Q.

Existential Quantification Contradictories Revisited

No humans are immortal.
Therefore all humans are mortal.

Let Hx represent x is human.
Let Mx represent x is mortal.

~∃(x)Hx~Mx
∴∀(x)Hx→Mx

~[∃(x)Hx~Mx]	
~[(Ha~Ma)]	E.I.
~Ha∨Ma	De M.
Ha→Ma	M.I.
∃(x)Hx→Mx	E.Q.

Not what we set out to prove.

Existential Quantification Contradictories Revisited

No humans are immortal.
Therefore all humans are mortal.

Let Hx represent x is human.
Let Mx represent x is mortal.

$\sim\exists(x)Hx{\sim}Mx$
$\therefore\forall(x)Hx\rightarrow Mx$

$\sim\exists(x)Hx{\sim}Mx$	
$\forall(x){\sim}(Hx{\sim}Mx)$	Contradictory
${\sim}(Hy{\sim}My)$	U.I.
${\sim}Hy\vee My$	De M.
$Hy\rightarrow My$	M.I.
$\forall(x)Hx\rightarrow Mx$	U.Q.

Working With Predicates and Quantifiers

- Define your variables and predicates.
- Transform language into predicates and quantified predicates.
- Use Contradictory transformations where necessary.
- Instantiate where needed.
- Try a forward proof.
- If that fails, try proof by contradiction.
- If that fails, test for invalidity.
- If all else fails, put everything into a truth table and inspect the results.
- Where appropriate, use quantification to generalize the results.

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

Let Px represent a person.
Let Cx represent x exercises self-control.
Let Tx represent x keeps their temper.
Let Jx represent x is a judge.

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

$\forall(x)PxCx \rightarrow Tx$
$\exists(x)PxJx{\sim}Tx$

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

$\forall(x)PxCx \rightarrow Tx$
$\exists(x)PxJx{\sim}Tx$

Simplify by defining your variable as a subset of ‘everything in the universe.’

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

Let x be a person.

Let Cx represent x exercises self-control.
Let Tx represent x keeps their temper.
Let Jx represent x is a judge.

Examples

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.

$\forall(x)Cx \rightarrow Tx$
$\exists(x)Jx{\sim}Tx$

Let j represent a judge who loses his temper.

Examples

∀(x)Cx→Tx ∃(x)Jx~Tx	Assume j represents Judy, a judge who does not keep her temper.
Jj~Tj	Existential Instantiation
~Tj	Simplification
Jj	Simplification
Cj→Tj	Universal Instantiation
∴~Cj	Modus Tollens
Jj~Cj	Conjunction
∴∃(x)Jx~Cx	Existential Generalization

Some judges lack self-control.

Examples

All humans are mortal.
Socrates is human,
Therefore, Socrates is mortal.

Let x be a human.
$\forall(x)Mx$

Assume Socrates is human.

$\forall(x)Mx$	Let s be the human, Socrates.
Ms	U.I.

Example

“All diligent students are successful.
All ignorant students are unsuccessful.”

What can you conclude about diligent students?
What can you conclude about ignorant students?

Example

"All diligent students are successful.
All ignorant students are unsuccessful."

Let x be a student.
Let Dx represent x is diligent.
Let Ix represent x is ignorant.
Let Sx represent x is successful.

$\forall(x)Dx \rightarrow Sx$
$\forall(x)Ix \rightarrow {\sim}Sx$

Example

∀(x)Dx→Sx
∀(x)Ix→~Sx

Q: What can you conclude about diligent students?

∀(x)Dx→Sx ∀(x)Ix→~Sx Dy	Let y be any diligent student.
Dy→Sy	Universal Instantiation
∴Sy	Modus Ponens
Iy→~Sy	Universal Instantiation
∴~Iy	Modus Tollens
Dy~Iy	Conjunction
∴∀(x)Dx~Ix	Universal Instantiation

Diligent students are not ignorant.

Example

∀(x)Dx→Sx
∀(x)Ix→~Sx

Q: What can you conclude about ignorant students?

Ignorant students are not diligent

∀(x)Dx→Sx ∀(x)Ix→~Sx Iy	Let y be any ignorant student.
Iy→~Sy	Universal Instantiation
∴~Sy	Modus Ponens
Dy→Sy	Universal Instantiation
∴~Dy	Modus Tollens
~DyIy	Conjunction
∴∀(x)~DxIx	Universal Instantiation

Example

Some pigs are wild.
All pigs are fat.

Let x be a pig.

Let Wx mean x is wild.
Let Fx mean x is fat.

Example

∃(x)Wx
∀(x)Fx

∃(x)Wx ∀(x)Fx	Let y be any pig.
Wy	Existential Instantiation
Fy	Universal Instantiation
WyFy	Conjunction
∴ ∃(x)FxWx	Existential Quantification

Example

∃(x)Wx
∀(x)Fx

Best practice, don't use a generalized variable for Existential Instantiation.

∃(x)Wx ∀(x)Fx	Assume c represents Charlotte, a wild pig.
Wc	Existential Instantiation
Fc	Universal Instantiation
WcFc	Conjunction
∃(x)FxWx	Existential Quantification

Example

Babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

What can you say about babies?

Let x be any baby.

Dx means x is despised.
Ix means x is illogical.
Mx means x can manage a crocodile.

Example

∀(x)Ix ∀(x)[Ix→Dx] ∀(x)[Mx→~Dx]	Let y be any baby
Iy	Universal Instantiation
Iy→Dy	Universal Instantiation
∴Dy	Modus Ponens
My→~Dy	Universal Instantiation
∴~My	Modus Tollens
∴∀(x)~Mx	Babies can't manage Crocodiles.

Example

1. All writers who understand human nature are clever;
2. No one is a true poet unless he can stir the hearts of men;
3. Shakespeare wrote “Hamlet”;
4. No writer, who does not understand human nature, can stir the hearts of men;
5. None but a true poet could have written “Hamlet.”

Example

Assume x is a writer.

Let Ux mean x understands human nature.
Let Cx mean x is clever.
Let Px mean x is a poet.
Let Hx mean x wrote “Hamlet.”
Let Sx mean x stirs the hearts of men.

Example

1. ∀(x)Ux→Cx
2. ~∃(x)Px~Sx
3. Hs
4. ~∃(x)~UxSx
5. ~∃(x)Hx~Px

Example

1. ∀(x)Ux→Cx
2. ∀(x)~[Px~Sx]
3. Hs
4. ∀(x)~[~UxSx]
5. ∀(x)~[Hx~Px]

Example

∀(x)Ux→Cx ∀(x)~[Px~Sx] ∀(x)~[~UxSx] ∀(x)~[Hx~Px] Hs	
~[Hs~Ps]	U.I.
~Hs∨Ps	De.M.
∴Ps	Conj. Syl.
~[Ps~Ss]	U.I.

~Ps∨Ss	De.M.
∴Ss	Conj. Syl.
~[~UsSs]	U.I.
Us∨~Ss	De. M.
∴Us	Conj. Syl.
Us→Cs	U.I.
∴Cs	M.P.

Example

1. Every idea of mine, that cannot be expressed as a Syllogism, is really ridiculous;
2. None of my ideas about Bath-buns are worth writing down;
3. No idea of mine, that fails to come true, can be expressed as a Syllogism;
4. I never have any really ridiculous ideas, that I do not at once refer to my solicitor;
5. My dreams are all about Bath-buns;
6. I never refer to any idea of mine to my solicitor, unless it is worth writing down.

Example

Let x represent an idea of mine.

Let Sx represent x can be expressed as a syllogism.
Let Bx represent x is about Bath-buns.
Let Wx represent x is worth writing down.
Let Tx represent x comes true.
Let Rx represent x is ridiculous.
Let Lx represent x is referred to my solicitor.
Let Dx represent x is a dream.

Example

1. $\forall(x){\sim}Sx \to Rx$
2. $\forall(x)Bx \to {\sim}Wx$
3. $\forall(x){\sim}Tx \to {\sim}Sx$
4. $\forall(x)Rx \to Lx$
5. $\forall(x)Dx \to Bx$
6. $\forall(x)Lx \to Wx$

Example

∀(x)~Sx→Rx ∀(x)Bx→~Wx ∀(x)~Tx→~Sx ∀(x)Rx→Lx ∀(x)Dx→Bx ∀(x)Lx→Wx Dy	Let y represent any dream of mine.
Dy→By	U.I.
∴By	M.P.
By→~Wy	UI
∴~Wy	M.P.

Ly→Wy	U.I.
∴~Ly	M.T.
Ry→Ly	U.I.
∴~Ry	M.T.
~Sy→Ry	U.I.
∴Sy	M.T.
~Ty→~Sy	U.I.
∴~~Ty	M.T.
Ty	D.N.
DyTy	Conj.
∴∀(x)DxTx	U.G.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

Example

Let Fx represent x is a fig.
Let Gx represent x is a grape.
Let Hx represent x is healthful.
Let Ix represent x is illaudable.
Let Jx represent x is jejune.
Let Kx represent x is knurly.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx
∴∃(x)FxIx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx ∴∃(x)FxIx?	Assume f is a fig that is knurly.
Ff~Kf	E.I.
Ff	Simp.
(Ff∨Gf)→Hx	U.I.
∴Hf	M.P.
Hf→~If~Jf	U.I.
∴~If~Jf	M.P.
~If	Simp.
∴∃(x)FxKx~Ix	

This is not at all what we set out to prove.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx
∴∃(x)FxIx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx ∴∃(x)FxIx? Fy	Assume y is any fig.
(Fy∨Gy)→Hy	U.I.
∴Hy	M.P.
Hy→~Iy~Jx	U.I.
∴~Iy~Jy	M.P.
~Iy	U.I.
Fy~Iy	Conj
∴∀(x)Fx~Ix	Simp.
~∴∃(x)FxIx	Refute by contradiction?

This is not at all what we set out to prove.

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx	Assume g is jejune and knurly grape.
GgJgKg	E.I.
Jg	Simp.
(Fg∨Gg)→Hg	U.I.
∴Hg	M.P.
Hg→~Ig~Jg	U.I.
∴~Ig~Jg	M.P.
~Jg	Simp.
Jg~Jg	

Example

∀(x)(Fx∨Gx)→Hx
∀(x)Hx→~(Ix∨Jx)
∃(x)GxJxKx
∃(x)Fx~Kx

∀(x)(Fx∨Gx)→Hx ∀(x)Hx→~Ix~Jx ∃(x)GxJxKx ∃(x)Fx~Kx	Assume g is jejune and knurly grape.
GgJgKg	E.I.
Jg	Simp.
(Fg∨Gg)→Hg	U.I.
∴Hg	M.P.
Hg→~Ig~Jg	U.I.
∴~Ig~Jg	M.P.
~Jg	Simp.
Jg~Jg	

The problem is ill-formed. The premises cannot all be true.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. Therefore, some figs are illaudable.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- ~~Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.~~
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. Therefore, some figs are illaudable.

Example (Copi)

Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore, some figs are illaudable.

- ~~Grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly.~~
- Figs are healthful. Nothing healthful is either illaudable or jejune. Some figs are not knurly. ~~Therefore, some figs are illaudable.~~ Therefore, some figs are not illaudable, not jejune, and not knurly.

Explain the transformation and keep the original for reference.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

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Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

What is x?

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let x be a person.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let Cx be a person that is a criminal.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let Rx be a person that robbed the Russell mansion.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let Sx be a person who had an accomplice among the servants.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let Bx be a person who broke into the Russell mansion.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No criminal could be both an expert locksmith and a convincing actor. No one could rob the Russell mansion unless he fooled the guard. Therefore some criminal had an accomplice among the servants.

Let Dx be a person who smashed the door.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No criminal could be both an expert locksmith and a convincing actor. No one could rob the Russell mansion unless he fooled the guard. Therefore some criminal had an accomplice among the servants.

Let Px be a person who picked the lock.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No criminal could be both an expert locksmith and a convincing actor. No one could rob the Russell mansion unless he fooled the guard. Therefore some criminal had an accomplice among the servants.

Let Lx be a person who is an expert locksmith.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No criminal could be both an expert locksmith and a convincing actor. No one could rob the Russell mansion unless he fooled the guard. Therefore some criminal had an accomplice among the servants.

Let Hx be a person who is was heard breaking in.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No criminal could be both an expert locksmith and a convincing actor. No one could rob the Russell mansion unless he fooled the guard. Therefore some criminal had an accomplice among the servants.

Let Fx be a person who fooled the guard at the Russell mansion.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

Let Ax be a person who is a convincing actor.

Example

- Let Rx be a person who robbed the Russell mansion.
- Let Cx be a person who is a criminal.
- Let Sx be a person who had an accomplice among the servants at the Russell mansion.
- Let Bx be a person who broke into the Russell mansion.
- Let Dx be a person who smashed the door of the Russell mansion.
- Let Px be a person who picked the lock of the Russell mansion.
- Let Lx be a person who is an expert locksmith.
- Let Hx be a person who was heard breaking into the Russell mansion.
- Let Fx be a person who fooled the guard at the Russell mansion.
- Let Ax be a person who is a convincing actor.

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

$\exists(x)CxRx$

Final Example

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$\forall(x)Rx \rightarrow (Sx \vee Bx)$

Final Example

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$\forall(x)Px \rightarrow Lx$

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$\forall(x)Dx \rightarrow Hx$

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

$\sim\exists(x)Hx$

$\forall(x)\sim Hx$

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

$\forall(x)CxRxFx \rightarrow Ax$

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

$\sim\exists(x)Rx{\sim}Fx$

$\forall(x)Rx\to Fx$

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

~∃(x)CxLxAx
∀(x)~(CxLxAx)

Final Example

Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or to pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants.

$\therefore \exists (x)CxSx$

Final Example

1. ∃(x)CxRx
2. ∀(x)Rx→(Sx⋎Bx)
3. ∀(x)Bx→(Dx⋎Px)
4. ∀(x)Px→Lx
5. ∀(x)Dx→Hx
6. ∀(x)~Hx
7. ∀(x)CxRxFx→Ax
8. ∀(x)Rx→Fx
9. ∀(x)~(CxLxAx)

//∴∃(x)CxSx

Start with the existentially quantified proposition.

Final Example

	Let b represent the criminal who burgled the Russell's mansion.
CbRb	E.I. (1)
~Hb	U.I. (6)
Db→Hb	U.I. (5)
∴~Db	M.T.
Rb	Simp.
Rb→Fb	U.I. (8)
∴Fb	M.P.

CbRbFb→Ab	U.I. (7)
CbRbFb	Conj.
∴Ab	M.P.
~(CbLbAb)	U.I. (9)
~Cb⋎~Lb∨~Ab	De M.
∴~Lb	D.S.
Pb→Lb	U.I. (4)
∴~Pb	M.T.
~Db~Pb	Conj
~(Db∨Pb)	De M.
Bb→(Db⋎Pb)	U.I. (3)
∴~Bb	M.T.

Final Example

Rb→(Sb⋎Bb)	U.I. (2)
∴Sb⋎Bb	M.P.
∴Sb	C.S.
Cb	Simp.
CbSb	Conj.
∴ ∃ CxSx	E.G.

There is a criminal who had an accomplice among the servants.

Final Example - Proof by Contradiction

1. $\exists(x)CxRx$
2. $\forall(x)Rx \rightarrow (Sx \curlyvee Bx)$
3. $\forall(x)Bx \rightarrow (Dx \curlyvee Px)$
4. $\forall(x)Px \rightarrow Lx$
5. $\forall(x)Dx \rightarrow Hx$
6. $\forall(x){\sim}Hx$
7. $\forall(x)CxRxFx \rightarrow Ax$
8. $\forall(x)Rx \rightarrow Fx$
9. $\forall(x){\sim}(CxLxAx)$

   $// \therefore \exists(x)CxSx$

Final Example Take 2

~∃(x)CxSx	Assumption
∀(x)~(CxSx)	Contradictory
CbRb	E.I. (1)
~Hb	U.I. (6)
~(CbSb)	U.I. (Assumption)
~Cb⋎~Sb	De M.
Cb	Simp.
∴~Sb	M.T.
Rb→Sb⋎Bb	U.I. (2)
∴Sb⋎Bb	M.P.
∴Bb	C.S.
Bb→(Db⋎Pb)	U.I. (3)
∴Db⋎Pb	M.P.
Db→Hb	U.I. (5)
∴~Db	M.T.
∴Pb	C.S.
Pb→Lb	U.I. (4)
∴Lb	M.P.
Rb→Fb	U.I. (8)
Rb	Simp.
∴Fb	M.P.

Final Example

CbRbFb	Conj.
CbRbFb→Ab	U.I. (7)
∴Ab	M.P.
~(CbLbAb)	U.I. (9)
CbLbAb	Conj.
∴∃CxSx	Proof by Contradiction.

There is a criminal who had an accomplice among the servants.

Conclusion

Formal logic is a set of techniques and rules of inference used to assign logic values to propositions.

If a statement can’t be assigned a truth value, then the statement is not subject to formal logic.

- Identify the prepositions.
- Identify the quantifiers.
- Transform language into propositions.
- Verify validity.
- Use function/replacement rules/rules of inference to draw conclusions.

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