Comparing and Improving Quark Models for the Triply Bottom Baryon Spectrum

A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Science degree in Physics from the College of William and Mary

by

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Date: April 2013

Abstract

The energy spectrum of baryons containing three bottom quarks has been calculated theoretically by studies using nonperturbative and nonrelativistic, lattice Quantum Chromodynamics (lattice QCD) and by studies using potential models to account for the different interactions between the quarks. We use the results of the Lattice QCD study done by Stefan Meinel [1] as a starting point for our investigation, because it is a thorough and recent description of how values for the energy levels can be found. We hope to understand the discrepancies between his QCD results and the results of potential models which also have listed values of the energy spectrum, most notably the potential models used by Silvestre-Brac [2][1], and Roberts and Pervin [3]. In comparing the energy values for the first ten excited states of these potential models, we start with a harmonic oscillator potential, with terms added in to account for the short range (Coulomb), spin-spin, and spin-orbit interactions. From this comparison, we hope to learn why these potential models disagree, and which is preferable, based on the authors' use of these three interactions. We also hope to learn whether or not a potential model can reproduce the results attained by Meinel and if such a potential model gives insight into the behavior of triply bottom baryon system.[2]

[1] We use the AP2 potential model from [2] for this paper.

[2] We exclude a tensor interaction term because we accept that the tensor interaction is small based on [2] and [3] noting that this term is small enough to neglect, and [2] still produces reasonably close values to the lattice QCD results.

Research

1. Introduction

The splittings of the energy levels seen in the results of the lattice QCD and the two potential models are shown below. We wanted to examine the interactions used in each potential model to understand which interactions contribute to each splitting and try to correlate the energy values of the ten excited states of the potential model with those from the QCD results in order to label each excited state.

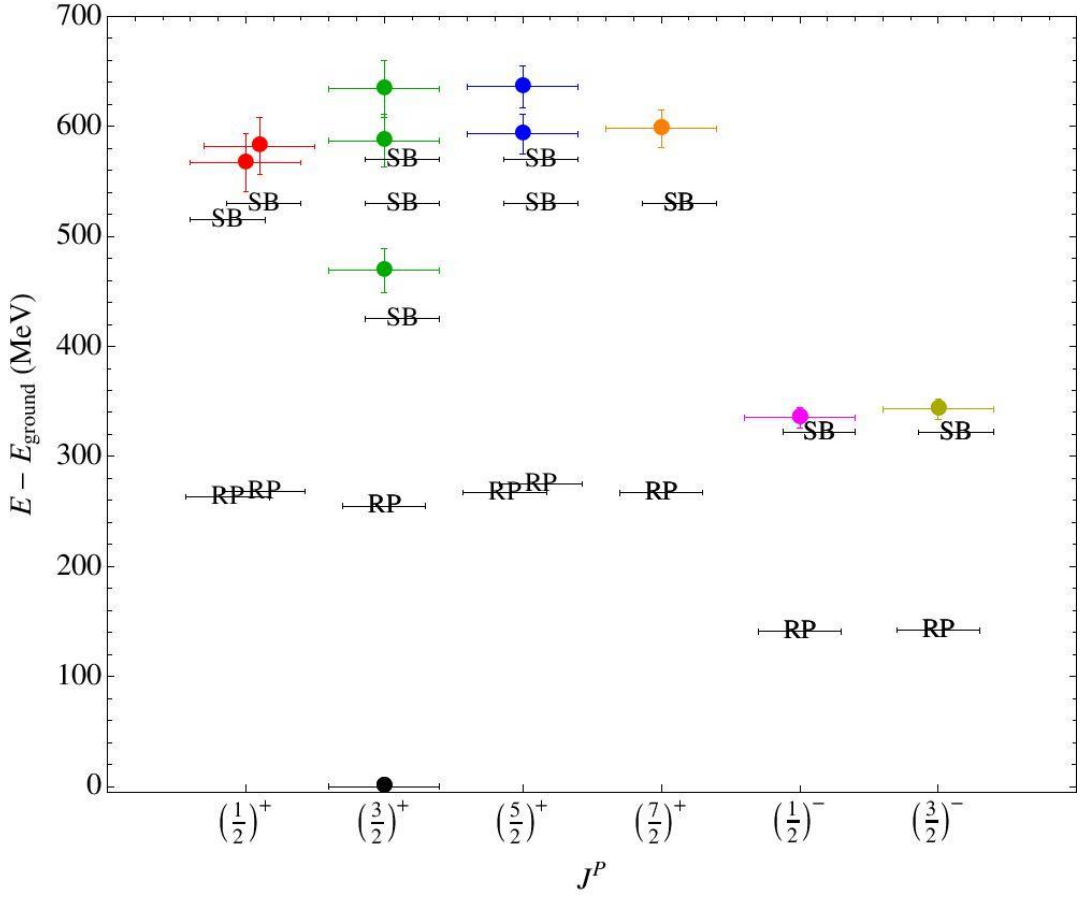

Figure 1: Comparison of the energy splittings from [1], [2] , and [3]. The colored dots are the QCD results, the initials "SB" are [2] and "RP" are the values from [3]. The energy splittings are given from the energy of the ground state; i.e. the ground state energy is the zero of the graph. The energy values are organized by total angular momentum J.

We start by identifying the wavefunctions of the triply bottom baryon, and then use these to perform first order energy corrections on terms included for each interaction. Based on the conclusions found in [2], we do not include the tensor interaction because it is negligibly small. We do include a Harmonic Oscillator potential as a starting point, then include a perturbing Hamiltonian for the short-range or Coulomb potential, the spin-spin interaction, and the spin-orbit interaction. Once these corrections are found, we use them to qualitatively compare the splittings between the different excited states.

2. Background and the Ground State

We began by studying how to describe the wavefunction of triply bottom baryon, and since baryons are *fermions*, and the wavefunction which describes the three baryons must be antisymmetric. This means that changing any two of the particles in the formalism will give the same wavefunction but with an opposite sign. The total wavefunction $|\Psi\rangle = |\Psi\ (1,2,3)\rangle$ is a combination of spatial, spin, flavor and color wavefunctions, written as:

$$|\Psi\ (1,2,3)\rangle = |spatial\rangle|spin\rangle|flavor\rangle|color\rangle \quad (1)$$

This function is antisymmetric in that:

$$|\Psi\ (1,2,3)\rangle = -|\Psi\ (2,1,3)\rangle \quad (2)$$

Each part of $|\Psi\rangle$ is a combination of the respective wavefunctions of each individual quark. Then, for example, the total wavefunction has a spatial sub-wavefunction $|spatial\rangle$, which is a combination of the spatial functions for each of the individual quarks, which could be written $|spatial\rangle = |S(1)S(2)S(3)\rangle$, where $S(n)$ represents the spatial state of the n^{th} particle. [3]

The total wavefunction is antisymmetric so the product of the sub-wavefunctions must be antisymmetric also. A sub-wavefunction is *symmetric* if switching any two of the wavefunctions of the individual quarks does not affect the total state; and a sub-wavefunction is *antisymmetric* if switching any two of the wavefunctions of the individual quarks gives the total state with the opposite sign. Often though, the sub-wavefunction does not fall into one of these categories, in which case we can make the product of two (or more) of the sub-wavefunctions a *mixed symmetry* state.

For the triply bottom baryon combination, two of the sub-wavefunctions are already determined. The color wavefunction is taken to be the totally antisymmetric color wavefunction $|color\rangle = \epsilon_{abc}$[4]. The flavor wavefunction is totally symmetric, since all three baryons are of bottom flavor, or $|flavor\rangle =$

[3] The first number in this Ket notation here describes the first baryon, the second describes the second baryon and the same holds for the third. This holds for the description of the other parts of the wavefunction as well.

[4] These colors describe how these baryons have strong force interactions, and these interactions are not pursued further than the affect these colors have on $|\Psi\rangle$, that is, by adding an antisymmetric component. The flavor component is treated in this way as well, and these two sub-wavefunctions are taken to be implicit in the total wavefunctions to follow, which are only products of spatial and spin sub-wavefunctions.

$|bbb\rangle$. This means that the product of the spatial and spin wavefunctions must be symmetric so that the total wavefunction, as a product of these parts, is antisymmetric.

To describe the symmetry of the spatial sub-wavefunctions, it is useful to relate the sub-wavefunction to the energy levels associated with each of the three baryons. In general, the energy level of each baryon can be described by a single index called the principal quantum number N. This number denotes the energy level E_N of a given system, and is an integer because the energy levels are *quantized,* that is, they have discrete values instead of a continuous range of values. The state E_0 of the system is the lowest energy level the system can obtain, and is called the *ground state*.

The principal quantum number is also used to determine the possible values for the orbital angular momentum of the system. The orbital angular momentum can take on values of $L = 0,1,2,3,\dots,N$, where $L = 0$ are defined as S-states, $L = 1$ are P-states, and $L = 2$ are D-states. These values are actually integer multiples of a value $\hbar = h/2\pi$, where h is Planck's constant, but it is convention to refer to orbital angular momentum in terms of the value of L rather than inserting this constant. Each energy level E_N of the total system (the three baryons together) determines the possible combinations of the orbital angular momentum of each baryon.

For the ground state $|\Psi\rangle = |\Psi_0\rangle$, $N = 0$ (and thus $L = 0$) and so there is no degeneracy, and each quark must have zero orbital angular momentum, meaning that each quark must be in an S-state, which is the state listed above as $|SSS\rangle$. For the first and second excited state $N = 1,2$, there are many possibilities, the spatial wavefunction could be any of the three types of symmetry, because switching the states of the baryons around can change the spatial sub-wavefunction in an asymmetrical manner. In order to produce the correct overall symmetry for states like these, it is necessary to find a product of the spin and spatial wavefunction.

The ground state will give insight into how to build a product of the spin and spatial wavefunctions that is symmetric. Since the spatial wavefunction is symmetric in the ground state, and the product of the spin and spatial wavefunctions must be symmetric, so the spin wavefunction must be symmetric as well.

A. Orbital Angular Momentum

Each quark has an orbital angular momentum $\boldsymbol{l}_\alpha$ and a spin[5] $\boldsymbol{s}_\alpha$ for a given state or total wavefunction; the total wavefunction also has a total orbital angular momentum $\boldsymbol{L}$ and a total spin $\boldsymbol{S}$[6]. We

[5] We take the conventional spin projection, that is, the projection of the spin along the z-axis. Also, $\alpha = 1,2,3$ for the three quarks.

[6] The relationship between the orbital angular momentum and spin of the particular quark and that of the total wavefunction is not a general relationship and depends on how these quantities add as vectors. Instead of taking all three orbital momentums, we will use a coordinate simplification to use only the orbital momentum projection along two directions, but continue using the spin projection of all three quarks.

can also describe the orbital angular momentum in terms of the parity of the system. Parity describes the transformation that occurs when the spatial wavefunction has the sign of its variable $\boldsymbol{r} = (r, \theta, \varphi)$. This is to say, parity describes the transformation of the wavefunction as $\boldsymbol{r}$ goes to $-\boldsymbol{r}$. Either the spatial wavefunction will remain the same or it may switch signs. Those spatial wavefunctions whose sign does not switch under the transformation[7] to –**r** are called *positive parity* states, while those that do have a sign switch are called *negative parity* states; those states which do not fall into either category do not have a definite parity. For the spatial portion of the ground state wavefunction, we have $|SSS\rangle$, and switching to $-\boldsymbol{r}$ will not affect these functions, so the ground state has positive parity, and the parity *p*, along with the total angular momentum $\boldsymbol{J}$ is written $\boldsymbol{J}^p = 3/2^+$. This result comes from the fact that these spatial states are spherical harmonic solutions to the Schrodinger equation for the potential models we consider here. These functions involve trigonometric functions (generally products of sines and cosines with an exponential function of φ) which help to determine how these three transformations affect the sign of the overall state. The ground state spherical harmonic does not have an angular dependence, and so does not affect the sign of the overall wavefunction, so the sign of the ground state spatial sub-wavefunction remains the same under this transformation, and so has positive parity.[8] Thus the ground state spatial sub-wavefunction has all three quarks in an S-state: $|\Psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)\rangle = |S(\boldsymbol{r}_1), S(\boldsymbol{r}_2), S(\boldsymbol{r}_3)\rangle$, which is completely symmetric and of positive parity.

B. Spin Angular Momentum

Each baryon has a spin which is an intrinsic angular momentum of the quark. Analogously to how the Earth rotates about the sun has angular momentum associated with its orbit (orbital angular momentum) and angular momentum associated with its rotation about its center of mass (called the spin angular momentum), each baryon has orbital angular momentum around their combined center of mass, and spin angular momentum. Although the spin of the baryons is not exactly analogous mathematically to the spin in classical mechanics, it does share one important property with the classical spin: the sum of the orbital angular momentum and the spin gives the total angular momentum of the system in both classical and quantum mechanics. The spin value $\boldsymbol{s}_\alpha$ for each quark is $\frac{1}{2}\hbar$, but the projection along the z-axis $\boldsymbol{m}_\alpha$ of each of the quarks can be either $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$.

The possible spin wavefunctions for triply bottom baryons can be enumerated [1]. These wavefunctions are combinations of the possible spin states of the three baryons; that is, we can enumerate

[7] Although this is referred to as one transformation, it is actually a collection of three transformations: $r \to -r$, $\theta \to \pi - \theta$, and $\varphi \to 2\pi - \varphi$.

[8] In general, parity flips at the next index of *N*; so the ground state has positive parity, the 1st excited state has negative parity, and the 2nd excited state has positive parity, and so on.

the possible combinations of the three baryons by varying which is baryon has a spin projection upwards or downwards along the z-axis (i.e. spin up or spin down) and then combine these into spin states. The possible combinations of the three baryon spin projections are:

$$|\boldsymbol{m}_1\boldsymbol{m}_2\boldsymbol{m}_3\rangle \in \{|\uparrow\uparrow\uparrow\rangle,\ |\uparrow\uparrow\downarrow\rangle,\ |\uparrow\downarrow\uparrow\rangle,\ |\downarrow\uparrow\uparrow\rangle,\ |\downarrow\downarrow\uparrow\rangle,\ |\downarrow\uparrow\downarrow\rangle,\ |\uparrow\downarrow\downarrow\rangle,\ |\downarrow\downarrow\downarrow\rangle\}$$

These spin combinations can then be combined into a number of symmetric spin sub-wavefunctions or spin states $|\boldsymbol{S},\boldsymbol{m}_s\rangle$[9]:

$$\begin{gathered}|\boldsymbol{S},\boldsymbol{m}_s\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle = |\uparrow\uparrow\uparrow\rangle \\ \left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\,|\uparrow\uparrow\downarrow\rangle + \,|\uparrow\downarrow\uparrow\rangle + \,|\downarrow\uparrow\uparrow\rangle) \\ \left|\frac{3}{2},\frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\,|\downarrow\uparrow\uparrow\rangle + \,|\downarrow\uparrow\downarrow\rangle + \,|\uparrow\downarrow\downarrow\rangle) \\ \left|\frac{3}{2},\frac{-3}{2}\right\rangle = \,|\downarrow\downarrow\downarrow\rangle\end{gathered} \tag{3}$$

In (3), the coefficients in the $\left|\frac{3}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2},\frac{-1}{2}\right\rangle$ states are to ensure each state is properly normalized and that the spin states obtain the proper projection along the z-axis. The square of the coefficient in from of each of the possible combinations, that is, the square of the *modulus*, is the probability that the baryons will be found in that particular combination. For instance, because each of these two states, $\left|\frac{3}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2},\frac{-1}{2}\right\rangle$, involves three terms, and the coefficient squared of each term is $1/3$, this means that any one of the three combinations is equally possible to obtain if the spin projection along the z-axis was to be measured for the total system.

The total spin of the system could be, $S = 1/2$, instead of $S = 3/2$, in which case we have other possible spin states. The spin combinations can form mixed-symmetric and mixed-antisymmetric spin states for $S = 1/2$. These are mixed states which are symmetric or anti-symmetric when interchanging the first and second particles, but not when interchanging the first and third, or second and third. This means the first two particles of the three particle state, when interchanged, will produce either a symmetric or antisymmetric wavefunction, while interchanging other pairs of the particles may not produce a state with a definite symmetry. The mixed symmetric spin states are:

$$\begin{gathered}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{MS} = \frac{1}{\sqrt{6}}(\,|\uparrow\downarrow\uparrow\rangle + \,|\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle) \\ \left|\frac{1}{2},\frac{-1}{2}\right\rangle_{MS} = \frac{1}{\sqrt{6}}(\,|\downarrow\uparrow\downarrow\rangle + \,|\uparrow\downarrow\downarrow\rangle - 2\,|\downarrow\downarrow\uparrow\rangle)\end{gathered} \tag{4}$$

[9] This notation gives the total spin of the state $\boldsymbol{S}$ and the total projection along the z-axis of that state $\boldsymbol{m}_s$.

The mixed-antisymmetric states are:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{MA} = \frac{1}{\sqrt{2}}(\,|\uparrow\downarrow\uparrow\rangle - \,|\downarrow\uparrow\uparrow\rangle)$$
$$\left|\frac{1}{2},\frac{-1}{2}\right\rangle_{MA} = \frac{1}{\sqrt{2}}(\,|\downarrow\uparrow\downarrow\rangle - \,|\uparrow\downarrow\downarrow\rangle) \quad (5)$$

As mentioned, the sum of the spin and the orbital angular momentum gives the total angular momentum $\boldsymbol{J}$ of the system:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \geq 0 \quad (6)$$

Each of the individual quarks may be either spin up or spin down, but for the spin sub-wavefunction to be symmetric, as is required of the ground state (since the spatial sub-wavefunction will be symmetric) all three must be spin up, which gives the ground state a total angular momentum of $\boldsymbol{J} = \frac{3}{2}$. So the product of the spin and spatial sub-wavefunctions for the ground state is:

$$|\Psi_0\rangle = |S(\boldsymbol{r}_1), S(\boldsymbol{r}_2), S(\boldsymbol{r}_3)\rangle \left|\frac{3}{2},\frac{3}{2}\right\rangle.$$

3. First Excited State

A. Orbital Angular Momentum

The first excited state $|\Psi_1\rangle$ has $N = 1$, and so the orbital angular momentum of the system could be one of two values $\boldsymbol{L} \in \{1, 0\}$. If $\boldsymbol{L} = 1$, then two of the baryons would need to be in S-states while one would be in a P-state, or for $\boldsymbol{L} = 0$, we have the same symmetric state $|S(\boldsymbol{r}_1), S(\boldsymbol{r}_2), S(\boldsymbol{r}_3)\rangle$ as for the ground state. We will show that, for $\boldsymbol{L} = 1$, there are no possible completely symmetric or completely antisymmetric spatial sub-wavefunctions for this case, and then show that two mixed symmetric states do exist.

We show that there is no completely symmetric spatial sub-wavefunction for $\boldsymbol{L} = 1$ first excited state. As mentioned above, these spatial states are products of a radial function and spherical harmonic functions. Here we consider the spherical harmonics, which we write as $|1,\ m_L\rangle = Y_{1,m_L}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$[10]. In this notation, the spherical harmonic Y_{1,m_L} is a function of the position vectors of the three particles, but this

[10] In this notation, m is the third principle quantum number, and $|\boldsymbol{L}| \geq m$, not to be confused with the projection of the total spin $\boldsymbol{m}_s$, or the three projections of the spin for each quark $\boldsymbol{m}_\alpha$.

may be written in terms of the momentum[11] $\boldsymbol{P}$ of the system as $|1,\ m_L\rangle = Y_{1,m_L} f(P^2)$, where $Y_{1,m}$ is still a function of the two angles θ and φ, but $f(P^2)$ is a function denoting the dependence of the state on the magnitude of the momentum squared. The spherical harmonics can related by ladder operators, which are constructed using cross products of $\boldsymbol{P}$ with the momentum operators $P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, P_y = \frac{\hbar}{i}\frac{\partial}{\partial y}, P_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$, which give the projection of the total momentum along the three Cartesian axes when applied to a given state $|1,\ m_L\rangle$. The state $|1,\ 0\rangle$ can be given by applying the following operator to a state dependent solely on the magnitude of the radius:

$$D_0 = \frac{-i}{\sqrt{2}} P_z = \frac{-\hbar}{\sqrt{2}}\frac{\partial}{\partial z}, \tag{7}$$

and the states one level higher or one level lower are given by the operators:

$$D_{\pm 1} = \pm\frac{i}{2}\left(P_x \pm iP_y\right) = \pm\frac{1}{2}\left(\hbar\frac{\partial}{\partial x} \pm i\hbar\frac{\partial}{\partial y}\right). \tag{8}$$

The possible spherical harmonics for the system are with $m_L = 1,0,-1$, since $|m_L| \leq L = 1$:

$$|1,\ m_L\rangle = \begin{cases} -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\, f(P^2) \ \text{ for } m_L = -1 \\ \sqrt{\frac{3}{4\pi}}\cos\theta\, f(P^2) \ \text{ for } m_L = 0 \\ \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\, f(P^2) \ \text{for } m_L = 1 \end{cases} = \begin{cases} -\sqrt{\frac{3}{4\pi}}\frac{1}{\sqrt{2}}\, f(P^2)(P_x + iP_y)/|P| \\ \sqrt{\frac{3}{4\pi}}P_z \\ \sqrt{\frac{3}{4\pi}}\frac{1}{\sqrt{2}}\, f(P^2)(P_x - iP_y)/|P| \end{cases} =$$

$$\sqrt{\frac{3}{4\pi}}\, f(P^2) P_{m_L}\frac{1}{|P|} = P_{m_L}\, g(P^2) \tag{9}$$

Where, in (6), $g(P^2)$ is a function of P^2 which contains the other constants, and P_{m_L} is the projection operator corresponding to the value of m_L. If we consider the completely symmetric state:

$$|1,\ m_L\rangle = \frac{1}{\sqrt{3}}(|PSS\rangle + |SPS\rangle + |SSP\rangle)$$

For $m_L = 0$, we have:

$$|1,\ 0\rangle = \frac{1}{\sqrt{3}}(|PSS\rangle + |SPS\rangle + |SSP\rangle) = P_0\, g(p^2) = \frac{1}{\sqrt{3}}(P_{1,z} + P_{2,z} + P_{3,z})\, g(P^2)$$

Where $\frac{1}{\sqrt{3}}$ is the normalization constant, and $P_{1,z}, P_{2,z}, P_{3,z}$ are the projections of the momentum along the z axis for each of the states combined in the symmetric spatial state. For this equation to be applicable, it must be able to handle the fact these particles are travelling very fast, and so must be invariant under changing reference frames. In particular, this equation must hold in the reference frame of the center of mass of the system. However, about the center of mass, the total momentum of the system is zero, so the sum of the projections of the momentum along the z axis $(P_{1,z} + P_{2,z} + P_{3,z})$ must also be zero, so our

[11] Working with the spatial states written in terms of momentum is conventionally referred to as "working in momentum space" and writing the states in terms of the radii as "working in position space."

spatial wavefunction goes to zero at $m_L = 0$ for a symmetric wavefunction, which cannot be the case, so the completely symmetric case cannot occur.

To see that there are no completely anti-symmetric spatial states, we look at the possible states that have one baryon in a P-state while the other two are in S-states. The possible states are the three used in showing the symmetric case is not possible: $|PSS\rangle, |SPS\rangle, |SSP\rangle$. Meinel notes that a completely anti-symmetric combination of these three states does not exist, and this can be verified by inspection, through changing the signs and coefficients of a combination of these states.

B. Spatial-Spin Product Sub-Wavefunctions

This leaves mixed symmetry spatial states, which require mixed symmetry spin states to make sure the total wavefunction of the system is antisymmetric. There is one mixed-symmetric state and one mixed-antisymmetric state, and both work for $m_L = 1$ and $m_L = -1$: [12]

$$|1\ \ m_L\rangle_{MS} = \frac{1}{\sqrt{6}}(2|SSP\rangle - |PSS\rangle - |SPS\rangle)$$
$$|1, m_L\rangle_{MA} = \frac{1}{\sqrt{2}}(|PSS\rangle - |SPS\rangle) \tag{10}$$

These states can be combined with the mixed symmetry spin states, which have a total spin of $S = \frac{1}{2}$, to produce the anti-symmetric total wavefunction required for the system of three baryons. Then we can construct the following four spatial-spin products:

$$|1,\ \ m_L\rangle_{MS}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_{MS};\ |1,\ \ m_L\rangle_{MS}\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle_{MS};\ |1,\ \ m_L\rangle_{MA}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_{MA};\ |1,\ m_L\rangle_{MA}\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle_{MA}.$$

These products can be combined to form a symmetric spatial-spin state:

$$\left|\boldsymbol{J},\ \ m_J\right\rangle = \sum_{m_1,m_2} \frac{C^{J,1,1/2}_{m_J,m_L,m_S}}{\sqrt{2}}\left(|1,\ \ m_L\rangle_{MS}\left|\frac{1}{2},m_S\right\rangle_{MS} + |1,\ \ m_L\rangle_{MA}\left|\frac{1}{2},m_S\right\rangle_{MA}\right)$$

Where $m_L = -1,1$, and $m_S = \frac{1}{2}, -\frac{1}{2}$, and $C^{J,1,1/2}_{m_J,m_L,m_S}$ is a notation for the correct Clebsch-Gordan coefficient for the values of m_L and m_S. The possible total angular momentum values for this state are:

$$\boldsymbol{J} = L + S = 1 \pm \tfrac{1}{2} = \begin{cases} \frac{3}{2} \text{ for } m_2 = 1/2 \\ \frac{1}{2} \text{ for } m_2 = -1/2 \end{cases} \tag{11}$$

[12] The subscripts here stand for Mixed Symmetric and Mixed Anti-symmetric, and we use this notation throughout.

As mentioned, the parity for the $N = 1$ energy level is negative, so for the 1st excited state, there are two values of the total angular momentum $\boldsymbol{J}^p = \frac{1}{2}^-, \frac{3}{2}^-$.

4. Second Excited State

For the second excited state, $N = 2$, so $L = 2,1,0$. Meinel shows that there are four possible spatial sub-wavefunctions which can be combined with the spin sub-wavefunctions to form the antisymmetric total wavefunction needed for this three baryon system. There are completely antisymmetric and completely symmetric spatial states, because the sum of the projections of the momenta mentioned in considering the first excited state becomes a sum of the square of the momenta for these D-states. The spatial wavefunction for D-states can be formed by taking the product of P-state spatial sub-wavefunctions [1][13]:

$$
\begin{aligned}
|L, m_L\rangle_S^{[2]} &= \sum_{m_1,m_2} C_{[2]}(+|1,\ m_L\rangle_{MS}|1,\ m_S\rangle_{MS} + |1,\ m_L\rangle_{MA}|1,\ m_S\rangle_{MA}) \\
|L, m_L\rangle_{MS}^{[2]} &= \sum_{m_1,m_2} C_{[2]}(-|1,\ m_L\rangle_{MS}|1,\ m_S\rangle_{MS} + |1,\ m_L\rangle_{MA}|1,\ m_S\rangle_{MA}) \\
|L, m_L\rangle_{MA}^{[2]} &= \sum_{m_1,m_2} C_{[2]}(+|1,\ m_L\rangle_{MS}|1,\ m_S\rangle_{MA} + |1,\ m_L\rangle_{MA}|1,\ m_S\rangle_{MS}) \\
|1, m_L\rangle_A^{[2]} &= \sum_{m_1,m_2} C_{[2]}(+|1,\ m_L\rangle_{MS}|1,\ m_S\rangle_{MA} - |1,\ m_L\rangle_{MA}|1,\ m_S\rangle_{MS})
\end{aligned}
\tag{12}
$$

Note that the first three combinations give either $L = 0$ or $L = 2$, while the last restricted to $L = 1$. These are then combined with the spin states from equations (3), (4) and (5), to produce the proper symmetry of the spatial and spin sub-wavefunctions [1]. We note that there is no completely anti-symmetric spin state combine with $|1, m\rangle_A^{[2]}$, so this spatial state is excluded, while the other three can be combined with spin states to produce the following four states that satisfy the antisymmetric requirement of the total wavefunction:

$$
\left|J, m_J, L = 2, m_L = 0, S = \frac{3}{2}, m_S = m\right\rangle = |0,0\rangle_S \left|\frac{3}{2}, m\right\rangle \tag{13}
$$

[13] The additional + sign is to emphasize the permutation of the sign to create the different symmetries, and $C_{[2]}$ are the correct Clebsch-Gordan coefficients for the sum. Here the subscripts on the spatial state ket notation are Symmetric, Anti-symmetric, and Mixed Symmetric and Mixed Anti-symmetric as above. The superscript is differentiate between the spatial states of the D-states ($N = 2$) from the P-states ($N = 1$) which are combined to form the D-states. Also $m = m_1 + m_2$, the sum of the projections of the momenta along the z axis, which we allow to vary in these states.

$$\left|J, m_J, L=2, m_L=0, S=\frac{1}{2}, m_S=m\right\rangle_m = \frac{1}{\sqrt{2}}\left(|0,0\rangle_{MS}\left|\frac{1}{2}, m\right\rangle_{MS} + |0,0\rangle_{MA}\left|\frac{1}{2}, m\right\rangle_{MA}\right)$$

$$\left|J, m_J, L=2, m_L=2, S=\frac{1}{2}, m_S=m\right\rangle = \frac{1}{\sqrt{2}}\sum_{m_1,m_2} C \quad \left(|2,m_1\rangle_{MS}\left|\frac{1}{2}, m_2\right\rangle_{MS} + |2,m_1\rangle_{MA}\left|\frac{1}{2}, m_2\right\rangle_{MA}\right)$$

$$\left|J, m_J, L=2, m_L=2, S=\tfrac{3}{2}, m_S=m\right\rangle = \textstyle\sum_{m_1,m_2} C \quad |2,m_1\rangle_S \left|\frac{3}{2}, m_2\right\rangle.$$

For the first product in (12), $\boldsymbol{J^p = L + S} = 0 + \frac{3}{2} = \frac{3^+}{2}$, since the state $\left|S=\frac{3}{2}, m_S=m\right\rangle$, must have $m = \frac{3}{2}$. For the second product, $\boldsymbol{J^p = L + S} = 0 + \frac{1}{2} = \frac{1^+}{2}$, since $\boldsymbol{J}$ must be greater than zero. For the third product, there are two values since there are two values for $m = \pm\frac{1}{2}$, then $\boldsymbol{J^p} = 2 \pm \frac{1}{2} = \frac{3^+}{2}, \frac{5^+}{2}$. For the last product, the values of m are: $-\frac{3}{2}, -\frac{1}{2}.\frac{1}{2}.\frac{3}{2}$, and these give $\boldsymbol{J^p} = \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2}, \frac{7^+}{2}$.

5. Energy State Summary

From the first and second excited states, we find ten possible values for the total angular momentum, but we use the eight values for the angular momentum produced by the second excited state for comparison between the potential models and the lattice QCD values. The graph of the energy levels given by Meinel [1] show how these eight levels split due to the other interactions. We reproduce this splitting qualitatively by including terms in our potential model, following the terms given by [2] and [3], which account for the short range, spin-spin, and spin-orbit interactions. In order to assess how each of these interactions affect the energy levels, we used 1st order perturbation theory to calculate the energy correction of the spin-spin and spin-orbit interaction, and used these results to model how these degenerate states split.

Table 1: Total Momentum of First and Second Excited States

N=1	$L=1$	$S=\frac{1}{2}$	$J^p=\frac{1^-}{2},\frac{3^-}{2}$
$N=2$	$L=0$	$S=\frac{1}{2}$	$J^p=\frac{1^+}{2}$
		$S=\frac{3}{2}$	$J^p=\frac{3^+}{2}$
	$L=2$	$S=\frac{1}{2}$	$J^p=\frac{3^+}{2},\frac{5^+}{2}$
		$S=\frac{3}{2}$	$J^p=\frac{1^+}{2},\frac{3^+}{2},\frac{5^+}{2},\frac{7^+}{2}$

6. Harmonic Oscillator Potential

Once we found the possible spatial-spin product sub-wavefunctions, we looked at the two potential models to see which of the two produced energy values most similar to [1] which was done by Silvestre-Brac [2]. We used the Roberts and Pervin results [3] to compare and constrast the two potential models to see where clear differences were, and how these differences might affect their results. In following [2], we used a Harmonic Oscillator potential with a short range perturbation, and then added terms for interactions that should contribute to the energy splittings. We wrote the Hamiltonian for this setup as:

$$H_0=\frac{-1}{2M}(P_1^2+P_2^2+P_3^2)+\frac{1}{2}k(\boldsymbol{r}_1-\boldsymbol{r}_2)^2+\frac{1}{2}k(\boldsymbol{r}_1-\boldsymbol{r}_3)^2+\frac{1}{2}k(\boldsymbol{r}_2-\boldsymbol{r}_3)^2$$

Where $M=m_b$ is the mass of the bottom baryons, and k is a spring constant. We then used three relative coordinates:

$$\begin{aligned}\boldsymbol{R}&=\frac{1}{3}(\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3)\\ \boldsymbol{\lambda}&=\sqrt{\frac{2}{3}}\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}-\boldsymbol{r}_3\right)\\ \boldsymbol{\rho}&=\frac{1}{\sqrt{2}}(\boldsymbol{r}_1-\boldsymbol{r}_2)\end{aligned}\qquad(14)$$

In order to rewrite the Hamiltonian in terms of the momentum of the three quarks. Using the Hamiltonian rewritten in terms of momentum, we can choose to work in the reference frame of the center of mass of the system, in which the net momentum about the center of mass is zero; this eliminates $\boldsymbol{R}$, which reduces

this three body problem to a two body problem with a momenta in the direction of $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$. The Hamiltonian can then be written in terms of a $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ part, so $H_0 = H_{\boldsymbol{\rho}} + H_{\boldsymbol{\lambda}}$, where:

$$H_{\mathbf{x}} = \frac{1}{2M} P_x^2 + \frac{1}{2} k' x^2$$

For $x = \boldsymbol{\lambda}, \boldsymbol{\rho}$, and $k' = 3k$ and $M = m_b$. We can then use this Hamiltonian to solve the Schrodinger equation, which was previously done by Isgur and Karl [4]. We use their solutions for the symmetric, antisymmetric and mixed symmetry spatial sub-wavefunctions, and then combine them with the spin sub-wavefunctions described above to find a product of the two states which give the proper symmetry of the overall wavefunction. Then we use these wavefunctions to calculate the energy corrections due to including other interactions.

$$\Psi_{00}^{S\prime} = \frac{1}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} \left(\rho^2 + \boldsymbol{\lambda}^2 + 3\alpha^{-2}\right) e^{\left[-\frac{1}{2}\alpha\left(\rho^2 + \boldsymbol{\lambda}^2\right)\right]}$$

$$\Psi_{00}^{\rho} = \frac{2}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} (\boldsymbol{\lambda} \cdot \boldsymbol{\rho})\, e^{\left[-\frac{1}{2}\alpha\left(\rho^2 + \boldsymbol{\lambda}^2\right)\right]} \tag{15}$$

$$\Psi_{00}^{\lambda} = \frac{1}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} \left(\rho^2 - \boldsymbol{\lambda}^2\right) e^{\left[-\frac{1}{2}\alpha\left(\rho^2 + \boldsymbol{\lambda}^2\right)\right]}$$

Where $\alpha = (k'M)^{\frac{1}{4}}$. These states are the symmetric and mixed symmetry spatial sub-wavefunctions for $L = 2$ and $N = 0$ states. Where the superscripts represent the symmetry of the state; the S superscript indicates complete symmetry, ρ or $\boldsymbol{\lambda}$ indicate which two coordinates are symmetric; that is, for the $\boldsymbol{\lambda}$ superscript, the first and second quarks are symmetric, and the third and the second are symmetric for the ρ superscript. Below are the sub-wavefunctions for $L = 2$, $N = 2$ states[14]:

$$\Psi_{22}^{S} = \frac{1}{2} \frac{\alpha^5}{\pi^{3/2}} (\rho_+^2 + \boldsymbol{\lambda}_+^2) e^{\left[-\frac{1}{2}\alpha\left(\rho^2 + \boldsymbol{\lambda}^2\right)\right]}$$

(16)

[14] For these states $v_+ = \frac{-1}{\sqrt{2}}\left(v_x + i v_y\right)$ for $v = \rho, \lambda$.Also, for Ψ_{22}^{ρ}, we believe there was a typo in [4], so we include a factor of one half in the exponential power, but including this value does not leave a normalized state.

$$\Psi_{22}^{\rho} = \frac{\alpha^5}{\pi^{3/2}}(\lambda_+\rho_+)\, e^{\left[-\frac{1}{2}\alpha\,(\rho^2+\boldsymbol{\lambda}^2)\right]}$$

$$\Psi_{22}^{\boldsymbol{\lambda}} = \frac{2}{\sqrt{3}}\frac{\alpha^5}{\pi^{3/2}}(\rho_+^2 - \boldsymbol{\lambda}_+^2)\, e^{\left[-\frac{1}{2}\alpha\,(\rho^2+\boldsymbol{\lambda}^2)\right]}$$

We determined from looking at a graph of the energy splitting in [1] that the splittings look, at a basic level, as either a harmonic oscillator potential or a Coulomb (or short range) potential. From this we sought to include a short range perturbation into the Hamiltonian given by $H_{Cou} = -\sum_{i,j}\frac{\sigma}{\boldsymbol{R}_{ij}}$, where $\boldsymbol{R}_{ij}$ is the distance between the i^{th} and j^{th} quark, and σ is the fine structure constant. Since we are looking at symmetric product states (i.e. the product of the spin and spatial sub-wavefunctions are symmetric) we can assume that $H_{Cou} = -3\frac{\sigma}{\boldsymbol{R}_{12}} = -3\sqrt{2}\sigma\left(\frac{1}{\rho}\right)$. Then, with our spatial sub-wavefunctions combined with the spin sub-wavefunctions described above, we can calculate the energy correction of this perturbative Hamiltonian. By constructing an integral table of various powers of ρ and λ we were able to find the energy corrections listed below.

State	**Energy Correction**[15]
$\left\|J=\frac{3}{2}, L=0\,, S=\frac{3}{2}\right\rangle$	$-33\frac{\alpha\sigma}{\sqrt{2\pi}}$
$\left\|J=\frac{1}{2}, L=0\,, S=\frac{1}{2}\right\rangle$	$-9.5\frac{\alpha\sigma}{\sqrt{2\pi}}$
$\left\|J\ \ , L=2\,, S=\frac{3}{2}\right\rangle$	$-4.6\frac{\alpha\sigma}{\sqrt{2\pi}}$
$\left\|J\ \ , L=2\,, S=\frac{1}{2}\right\rangle$	$-3.3\frac{\alpha\sigma}{\sqrt{2\pi}}$

These energy corrections qualitatively match the splittings found in [1]; one state, the $\left|J=\frac{3}{2}, L=0\,, S=\frac{3}{2}\right\rangle$ has an energy much lower than the other states, and the difference between the

[15] For the two last states, there are no specified total angular momentum values because these energy corrections hold for any of the possible total angular momentum values. For the last state, the wavefunction was not properly normalized which requires a factor of $\frac{1}{2}$ correction. With the correction, the larger value is the energy correction $-6.3\frac{\alpha\sigma}{\sqrt{2\pi}}$; without the correction, the smaller correction would occur, which is listed above.

$J = \frac{1}{2}$ states is produced. The main qualitative disparity is the way two states (one with $J = \frac{3}{2}$, another with $J = \frac{5}{2}$) have their energy raised slightly higher than the other states, which does not show up from just the harmonic oscillator and the Coulomb perturbation, so other interactions must be included.

7. Spin-Spin Interaction

To model the interaction of the spin of each baryon on the other two, we treat each of the interaction between particles similarly to how the spin of an electron interacts with the spin of a proton. The spin interaction of protons and electrons is based on the interactions between their magnetic moments μ_e and μ_p, since these are proportional to their respective spins. Then the spin interactions of a proton and an electron can be modeled as $H_{ss} = f(\boldsymbol{r})\overline{\mu_e} \cdot \overline{\mu_p} = g(\boldsymbol{r})\overline{s_e} \cdot \overline{s_p}$, where the function $g(\boldsymbol{r})$ depends on the relative distance of the proton and electron $\boldsymbol{r}$, and also contains the proportionality constants between the spin and the magnetic moments.

We then extend this to the three baryons by taking:

$$H_{ss} = \sum_{a<b}(\overline{s_a} \cdot \overline{s_b})\, C_{ss}(\mathbf{r}_{ab}),$$

Where $C_{ss}(\mathbf{r}_{ab})$, is a function of the relative distance between the three particles. If we take C_{ss} to be independent of the relative position, $C_{ss}(\mathbf{r}_{ab}) = C_{ss}$, we can take $H_{ss} = C_{ss}\sum_{a<b}(\overline{s_a} \cdot \overline{s_b})$. We can then calculate the expectation values for H_{ss} for a wavefunction $|\Psi\rangle$for the baryons:

$$\langle H_{ss}\rangle = \langle\Psi|H_{ss}|\Psi\rangle$$

$$= C_{ss}\langle\Psi|\left(\sum_{\alpha<\beta}\left(\overline{s_\alpha} \cdot \overline{s_\beta}\right)\right)|\Psi\rangle$$

Then note that for the total spin operator $\overline{s_T}$ and the spin squared operator for each individual particle s_α^2 that:

$$\overline{s_T{}^2} - \sum_{\alpha=1}^{3} s_\alpha^2 = \overline{s_1} \cdot \overline{s_2} + \overline{s_2} \cdot \overline{s_3} + \overline{s_1} \cdot \overline{s_3} = \sum_{\alpha<\beta}\left(\overline{s_\alpha} \cdot \overline{s_\beta}\right).$$

We then replace this is the above equation:

$$\langle H_{ss}\rangle = C_{ss}\langle\Psi|\left(\overline{s_T{}^2} - \sum_{\alpha=1}^{3} s_\alpha^2\right)|\Psi\rangle.$$

Then, for any wavefunction: $s_a^2 = s_a(s_a + 1) = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}$ since the spin of every baryon is always $\frac{1}{2}$, so this expectation value is only a function of the total spin eigenvalue of $|\Psi\rangle$. We only have two values for the total spin: $\frac{1}{2}$ or $\frac{3}{2}$. For the states with total spin $\frac{1}{2}$, $\langle H_{ss}\rangle = -\frac{3}{4}C_{ss}$ and for states with total spin $\frac{3}{2}$, ,

$\langle H_{ss}\rangle = \frac{3}{4}C_{ss}$. So a radially independent spin-spin interaction shifts some states up and some states down by the same value, which does break the degeneracy but not close to how the splittings are from [1]. This sort of interaction does not contribute qualitatively to the energy splittings found from the Coulomb potential, so we factored in a radial dependent spin-spin interaction using a 3-D Dirac delta function:

$$H_{ss} = A\sum_{i>j}\delta^3\left(\boldsymbol{r}_{ij}\right)\bar{s_i}\cdot\bar{s_j} = 3A\delta^3(\boldsymbol{r}_{12})\bar{s_1}\cdot\bar{s_2} = \frac{1}{2\sqrt{2}}\delta^3(\boldsymbol{\rho}). \quad (17)$$

When then performed the energy correction calculations on this perturbative Hamiltonian and got the energy corrections below.

State	Energy Correction
$\left\|J=\frac{3}{2}, L=0\ , S=\frac{3}{2}\right\rangle$	$\frac{39}{32}\frac{A}{\sqrt{2}}\frac{\alpha^3}{\pi^{3/2}}$
$\left\|J=\frac{1}{2}, L=0\ , S=\frac{3}{2}\right\rangle$	$\frac{15}{64}\frac{A}{\sqrt{2}}\frac{\alpha^3}{\pi^{3/2}}$
$\left\|J\ , L=2\ , S=\frac{3}{2}\right\rangle$	$\frac{3}{64}\frac{A}{\sqrt{2}}\frac{\alpha^3}{\pi^{3/2}}$
$\left\|J\ , L=2\ , S=\frac{1}{2}\right\rangle$	$\frac{1}{2}\frac{3}{64}\frac{A}{\sqrt{2}}\frac{\alpha^3}{\pi^{3/2}}$

We determined a value for the constant A by comparing this spin-spin interaction with the Hamiltonian of a hydrogen atom with the same spin-spin interaction and extending it to the three particle system. We found that $A = \frac{-8\pi}{3}\frac{e^2}{M}\sigma$. We also obtained a value for α by estimating a value for k' from the value used by light quark systems. We took $\omega = \sqrt{k'} = \sqrt{3k} = .36$ GeV, so $\alpha = (\omega^2 s^2) = 1.15$ GeV for a light quark mass s. Then to find a value of β around the 600 MeV energy levels we were expecting, we must have $\beta = .01$, or about 1% of the energy of the levels we are studying, which is very small. Even using these values for, say, the energy of the state $\frac{39}{32}\frac{A}{\sqrt{2}}\frac{\alpha^3}{\pi^{3/2}} \approx 77$ MeV, which is very small, so we concluded that the spin-spin interaction must not be large.

8. Spin Orbit Interaction

To include the spin orbit interaction, we use a radial dependent perturbative Hamiltonian:

$$H_{LS} = \sum_{i>j} \frac{D}{\boldsymbol{r}_{ij}^3}\left(\boldsymbol{L}_{ij} \cdot \boldsymbol{S}_{ij}\right) = \frac{3}{2\sqrt{2}}\frac{D}{\boldsymbol{\rho}^3}\left(\boldsymbol{L}_{\rho} \cdot \boldsymbol{S}_{\rho}\right) \quad (18)$$

Which we assume because of the symmetry of the spin and spatial sub-wavefunction symmetries. Using the expansion of $\boldsymbol{L}_{\rho} + \boldsymbol{S}_{\rho}$, we can rewrite $\boldsymbol{L}_{\rho} \cdot \boldsymbol{S}_{\rho} = \frac{1}{2}\left[\boldsymbol{J}_{\rho}^2 - \boldsymbol{L}_{\rho}^2 - \boldsymbol{S}_{\rho}^2\right]$. Using the fact that the baryons are indistinguishable and that $\boldsymbol{J}_{\rho}$ is the total angular momentum projected along the ρ axis, which is the combination of the angular momentum of two out of the three quarks, we can subtract off the spin of the third quark to get from the total angular momentum $\boldsymbol{J}$ to obtain a formula for $\boldsymbol{J}_{\rho}$:

$$\boldsymbol{J}_{\rho}^2 = (\boldsymbol{J} - \boldsymbol{S}_3)^2 = \boldsymbol{J}^2 - 2\boldsymbol{J} \cdot \boldsymbol{S}_3 - \boldsymbol{S}_3^2 = \boldsymbol{J}^2 - \frac{2}{3}\boldsymbol{J} \cdot \boldsymbol{S} - \boldsymbol{S}_3^2$$

We used this formula to compute the energy corrections for the $\left| J = \frac{7}{2}, L = 2, S = \frac{3}{2} \right\rangle$ state, and found an energy correction of $\langle H_{LS} \rangle = \frac{D}{5\sqrt{2}}\frac{\alpha^3}{\sqrt{\pi}}$. and for the $\left| J = \frac{1}{2}, L = 2, S = \frac{3}{2} \right\rangle$ state we found an energy correction of $\langle H_{LS} \rangle = -\frac{3D}{10\sqrt{2}}\frac{\alpha^3}{\sqrt{\pi}}$.. For the other two states (with $J = \frac{3}{2}, \frac{5}{2}$) and the states with $L = 0$ we need to rewrite $\boldsymbol{J}_{\rho}$ so that the quark momentums can be added as vectors instead of being assumed to be along the same axis (as we did for these two cases). In order to calculate $\boldsymbol{J}_{\rho}$ for the other states, we use:

$$\boldsymbol{L}_{\rho} \cdot \boldsymbol{S}_{\rho} = \frac{1}{2}\left[\frac{2}{3}\boldsymbol{J}^2 + \frac{1}{3}\boldsymbol{L}^2 - \frac{1}{3}\boldsymbol{S}^2 + \boldsymbol{S}_3^2 - \boldsymbol{L}_{\rho}^2 - \boldsymbol{S}_{\rho}^2\right]. \quad (19)$$

This formula may not be correct, so we do not have values for the energy correction due to the spin orbit interaction for the other states. However, we do have a formula to relate the constant D to A:

$$D = \frac{3}{16\pi}A,$$

So $D \approx \frac{1}{16}A$.

Conclusion

From our included interactions, our overall Hamiltonian looks like:

$$H_{total} = H_{\boldsymbol{\rho}} + H_{\boldsymbol{\lambda}} + H_{Cou} + H_{ss} + H_{LS} \quad (20)$$

Where the overall kinetic energy term has been removed because we worked from the center of mass reference frame. From the additional interactions we included, the most successful perturbation, that is, the perturbation that produced the most qualitatively accurate splittings is the Coulomb potential. This perturbation "pushed" the energy value of the $\left| L = 0, S = \frac{3}{2} \right\rangle$ state down further than any other state, and

also pushed the energy value of the $\left| L = 0 , S = \frac{1}{2} \right\rangle$ state down to create the smaller splitting seen between the two $J = \frac{1}{2}$ states. Although these two splittings are qualitatively produced by the Coulomb potential, this perturbation is not completely the cause for the splitting between the $J = \frac{3}{2}$ and $J = \frac{5}{2}$ states, which would be between the $\left| L = 2 , S = \frac{3}{2} \right\rangle$ and $\left| L = 2 , S = \frac{1}{2} \right\rangle$ states, and the complete splitting of these two levels must be in part due to the spin-dependent interaction terms.

We also determined that the spin-spin interaction should be a small effect by finding an estimate for α, which would correspond to the relatively small splittings seen in the QCD results. However, the estimate we found is much too small to match the splitting seen in the QCD, that is, to have one of the $J = \frac{3}{2}$ and one of the $J = \frac{5}{2}$ have higher energies than the other states with the same total angular momentum. The ratio between the two energy corrections for the $\left| L = 2 , S = \frac{3}{2} \right\rangle$ and $\left| L = 2 , S = \frac{1}{2} \right\rangle$ states is $\frac{1}{2}$ though so in future research, a better estimate for α may produce better energy correction values, and choosing the sign of α may contribute to labeling which states are "higher" and which are "lower" in terms of their energy values.

The potential models, then, do give some insight into the system; the effect of the Coulomb potential dominates the how the quarks interact, and the spin-spin and spin-orbit interactions should give rise to the additional splitting recorded by Meinel [1]. The Difference between the results of [2] and [3], then, seem to result from the choice of the parameters of the potentials; Silvestre-Brac chose a "flatter" short range interaction as $\gamma r^{2/3}$, whereas Roberts and Pervin chose a linear short range interaction, which may be the cause of the disparity in their results. The "flatness" of the potential used may be tied to the low estimate of α because of the mass dependence of the potential. For Monomial potentials (that is, potentials which are proportional to a fixed single power of the separation coordinate, in our case r, in coordinate space), there is a predictable energy splitting due to the mass of the quarks involved. The scaling of something between a pure linear and a pure harmonic oscillator potential is found in the QCD calculations. Silvestre-Brac, with a $\gamma r^{2/3}$ short range potential term, used something in between those two potentials, which could have approximated the mass scaling well enough to produce results closer to the QCD study. Roberts and Pervin used a linear potential, which may have skewed the mass scaling. Since this skewed scaling would be more noticeable with heavier particles, using bottom quarks may have resulted in their low energy values (since there values are less than the other two studies by about the mass of a bottom quark).

In future research then, it may be best to begin with a flatter potential, even if the energy spectrum must be calculated numerically because the QCD results suggest a potential in between the Harmonic

Oscillator and the Coulomb potential may best fit the system. In order to determine other insights a potential model could give, we would need to explore and possibly correct the estimate of α, and figure out the normalization for the $\left| J\ , L = 2\ , S = \frac{1}{2} \right\rangle$ wavefunction so to fully determine the energy corrections due to the spin-spin interaction.

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