Nonlocality in a Stochastic Approach to Quantum Mechanics and Quantum Field Theory

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Abstract

The conflict between general relativity and quantum mechanics is a longstanding problem in physics, and most of the proposed solutions require the existence of new physics at short distance scales. The random variations of stochastic mechanics could provide this new short-distance physics, but in this framework the nonlocality related to quantum entanglement could conflict with relativity. This paper reviews the basics of stochastic mechanics and the stochastic process for the ground state of the free scalar field. This is expanded on by calculating the stochastic process for a free scalar particle, the entangled state of 2 particles from separate scalar fields, and the ground state of the Dirac field. The inherent nonlocality of the theory is then addressed through the use of foliations of space-time, in which the dynamics of the fields are indexed by space-like hypersurfaces across which the fields evolve in a covariant manner. Additional comments on the use of discrete stochastic impulses and their relation to the Markov property and Lorentz invariance are also included.

Chapter 1

Introduction

The conflict between general relativity and quantum mechanics is a longstanding problem in physics, and most proposed solutions require the existence of new physics at short distances. This paper is motivated by an approach[1] which builds on Sakharov's formulation for induced gravity[2]. If quantum field theory emerges from a more fundamental framework that takes over at some short distance scale, then the presence of diffeomorphism invariance in the fundamental framework can give rise to gravitation as an emergent long distance interaction. Here the short distance scale at which the fundamental framework manifests plays the role of the regulator in Sakharov's theory that results in induced gravity. One potential candidate for this fundamental framework is Nelson's formulation of stochastic mechanics[3]. By allowing for short random jumps in the positions of particles, nonrelativistic quantum mechanics arises naturally as an emergent description. These random jumps provide a short distance physical mechanism that could act as the regulator needed for induced gravity. While stochastic mechanics has been shown to be consistent with quantum field theory in certain cases, this aspect of the theory has not been as thoroughly explored. Furthermore, the nonlocality associated with quantum entanglement could conflict with relativity. Given the central role of diffeomorphism invariance in the fundamental theory as a requirement for induced gravity, it is necessary to show that stochastic mechanics can exhibit this property. It is this problem of demonstrating diffeomorphism invariance which is the focus and motivation of this paper, with the potential of developing a theory of emergent gravity acting as a greater overall motivating factor.

Chapter 2

Stochastic Mechanics

While a full treatment of Nelson's stochastic mechanics can be found in a variety of other sources [3][4][5], I will provide here a basic overview, following the notation and general methodology of [4]. To begin with, any solution to the Schroedinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi \tag{2.1}$$

can be written in the form

$$\Psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\exp\left(\frac{i}{\hbar}S(\vec{x},t)\right), \tag{2.2}$$

where $\rho(\vec{x},t)$ and $S(\vec{x},t)$ are real. Introducing the velocity field

$$v(\vec{x},t) = \frac{1}{m}\nabla S(\vec{x},t) \tag{2.3}$$

allows the real and imaginary parts of the Schroedinger equation to be written in the form

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\vec{x}) - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0 \tag{2.4}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \tag{2.5}$$

which describes the hydrodynamical system called the Madelung fluid. The next step is to equate this description to a stochastic process. Due to the presence of random jumps in the time evolution of the system, it is necessary to define the following forward and backward time derivatives.

$$D_{(+)}f(\vec{x},t) = \lim_{\delta t \to 0^+} E\left(\frac{f(\vec{x}(t+\delta t), t+\delta t) - f(\vec{x}(t),t)}{\delta t}\right) \tag{2.6}$$

$$D_{(-)}f(\vec{x},t) = \lim_{\delta t \to 0^+} E\left(\frac{f(\vec{x}(t),t) - f(\vec{x}(t-\delta t), t-\delta t)}{\delta t}\right) \tag{2.7}$$

Here $E()$ denotes the ensemble average. For the stochastic differential equation

$$dq(t) = v_{(+)}(q(t),t)dt + dw(t) \tag{2.8}$$

describing the evolution of the position $q(t)$ of a particle, the following equations hold.

$$D_{(\pm)}q(\vec{x},t) = v_{(\pm)}(\vec{x},t) = \frac{1}{m}\nabla S_{(\pm)}(\vec{x},t) \tag{2.9}$$

$$S(\vec{x},t) = \frac{S_{(+)}(\vec{x},t) + S_{(-)}(\vec{x},t)}{2} \tag{2.10}$$

$$v(\vec{x},t) = \frac{v_{(+)}(\vec{x},t) + v_{(-)}(\vec{x},t)}{2} = \frac{1}{m}\nabla S \tag{2.11}$$

$$\delta S(\vec{x},t) = \frac{S_{(+)}(\vec{x},t) - S_{(-)}(\vec{x},t)}{2} = \frac{\hbar}{2}\ln\left(\rho(\vec{x},t)\right) \tag{2.12}$$

$$\delta v(\vec{x},t) = \frac{v_{(+)}(\vec{x},t) - v_{(-)}(\vec{x},t)}{2} = \frac{1}{m}\nabla(\delta S) \tag{2.13}$$

Here δS and δv are simply function names, and the full proof of these equations can be found in [4]. Equation (2.11) matches the velocity field, and the forward and backward Fokker-Planck equations for the probability density ρ provides the continuity equation (2.5). Equation (2.4) can be obtained by defining the acceleration as

$$a(\vec{x},t) = \frac{1}{2}(D_{(+)}D_{(-)} + D_{(-)}D_{(+)})q(\vec{x},t) \tag{2.14}$$

and assuming a conservative force

$$F(\vec{x},t) = ma(\vec{x},t) = -\nabla V(\vec{x}) \tag{2.15}$$

Plugging in and appropriately rearranging values for the forward and backward velocities and derivatives then provides Eq. (2.4), indicating that a system of particles undergoing stochastic motion can be described equivalently by the equations for the Madelung fluid. It is worth noting that S and ρ are introduced independently in the stochastic and Madelung fluid descriptions, and only later shown to correspond to the same function in both descriptions. In particular, since ρ is introduced as the probability density in the stochastic description and later equated with the amplitude of the wavefunction, stochastic mechanics explicitly predicts Born's rule.

2.1 Stochastic Field Theory

From here forward, we will work in natural units, where

$$\hbar = c = 1.$$

Guerra and Ruggiero define the stochastic process for the ground state of the free scalar field in the following way[6]. First, they identify the stochastic differential equation for the ground state of the nonrelativistic, one dimensional harmonic oscillator as

$$dq_0(t) = -\omega q_0(t)dt + dw(t) \tag{2.16}$$

They then note that since the momentum modes $\phi_{\vec{k}}$ for the ground state of the free scalar field satisfy the same harmonic oscillator equation with $\omega_{\vec{k}}^2 = m^2 + \vec{k}^2$, each momentum mode can be promoted to a stochastic process satisfying

$$d\phi_{\vec{k}}(t) = -\omega_{\vec{k}}\phi_{\vec{k}}(t)dt + dw_{\vec{k}}(t) \tag{2.17}$$

The momentum modes can then be collected through a Fourier transform to give the stochastic differential equation for the field $\phi(\vec{x})$ as

$$d\phi(\vec{x},t) = -\sqrt{-\nabla^2 + m^2}\phi(\vec{x},t)dt + dw(\vec{x},t) \tag{2.18}$$

It is worth mentioning here that evaluating the square root term requires taking its Taylor expansion, which in turn requires the value of the function to be known for all points in space. This introduces a nonlocal component that must be considered carefully, but is generally expected by the requirement of Bell's inequality that all hidden variable theories must be nonlocal.

2.2 Stochastic Field Theory for Entangled States

Let $\phi(\vec{x})$ be a real free scalar field with Hamiltonian

$$H = \int d^3x \frac{1}{2}(\pi_\phi^2 + (\nabla\phi)^2 + m^2\phi^2) \tag{2.19}$$

where $\pi_\phi = -i\frac{\partial}{\partial\phi} = -i\dot{\phi}$ is the canonical momentum. The Schroedinger wavefunctional $\Psi(\phi(\vec{x}), t)$ then satisfies

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{2}\int d^3x(-\frac{\partial^2}{\partial\phi^2} + (\nabla\phi)^2 + m^2\phi^2)\Psi. \tag{2.20}$$

Using the Fourier transform

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}, \tag{2.21}$$

this can be rewritten in terms of the momentum modes $\phi_{\vec{k}}$ so that $\Psi(\{\phi_{\vec{k}}\}, t)$ satisfies

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{\partial^2}{\partial\phi_{\vec{k}}\partial\phi_{-\vec{k}}} + \omega_{\vec{k}}^2\phi_{\vec{k}}\phi_{-\vec{k}}\right)\Psi. \tag{2.22}$$

Using the fact that $\phi_{\vec{k}} = \phi^*_{-\vec{k}}$ since $\phi(\vec{x})$ is real, and decomposing it into real and imaginary parts like $\phi_{\vec{k}} = \phi_R(\vec{k}) + i\phi_I(\vec{k})$, this equation can be rewritten again as

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\frac{\partial^2}{\partial\phi_R(\vec{k})^2} + \frac{\partial^2}{\partial\phi_I(\vec{k})^2} + \omega_{\vec{k}}^2\left(\phi_R(\vec{k})^2 + \phi_I(\vec{k})^2\right)\right]\Psi[\phi_R(\vec{k}), \phi_I(\vec{k}), t]. \tag{2.23}$$

By treating the integral as an infinite sum over the continuous independent values of $\vec{k}$ which arises from differentiating Ψ, and considering the product rule, it becomes clear that Ψ must be a product of harmonic oscillators for $\phi_R(\vec{k})$ and $\phi_I(\vec{k})$ over all independent values of $\vec{k}$. Letting $\{\phi_R(\vec{k}), \phi_I(\vec{k})\}$ denote the set of momentum modes corresponding to a given $\phi(\vec{x})$, the ground state is

$$\begin{aligned}
\Psi[\{\phi_R(\vec{k}), \phi_I(\vec{k})\}, t] &= \prod_{\vec{k} \text{ independent}} \psi_0(\phi_R(\vec{k}), t)\psi_0(\phi_I(\vec{k}), t) \\
&= \prod_{\vec{k} \text{ independent}} \sqrt{\frac{-m\omega_{\vec{k}}}{\pi}} e^{\frac{-m\omega_{\vec{k}}}{2}(\phi_R(\vec{k})^2 + \phi_I(\vec{k})^2) - i\omega_{\vec{k}} t} \\
&= A \exp\left(\int \frac{d^3k}{(2\pi)^3} \left[\frac{-m\omega_{\vec{k}}}{2}(\phi_R(\vec{k})^2 + \phi_I(\vec{k})^2) - i\omega_{\vec{k}} t\right]\right) \\
&= A \exp\left(\frac{-m}{2}\int \frac{d^3k}{(2\pi)^3} |\sqrt{\omega_{\vec{k}}}\phi_{\vec{k}}|^2\right) \exp\left(-i\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}} t\right) \\
&= A \exp\left(\frac{-m}{2}\int d^3x \sqrt{-\nabla^2 + m^2}\phi(\vec{x})^2\right) \exp\left(-i\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}} t\right)
\end{aligned} \tag{2.24}$$

where A is a normalization constant (here $A = \prod_{\vec{k} \text{ independent}} \sqrt{\frac{-m\omega_{\vec{k}}}{\pi}}$), the integral in line 3 comes from taking the sum over the continuous $\vec{k}$, and the Plancharel theorem is used between lines 4 and 5. The relevant values for the stochastic process are then

$$S = -\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}} t \tag{2.25}$$

$$\rho = A^2 \exp\left(-m\int d^3x \sqrt{-\nabla^2 + m^2}\phi(\vec{x})^2\right) \tag{2.26}$$

$$\delta S = \frac{1}{2}\ln(\rho) = \ln(A) - \frac{m}{2}\int d^3x \sqrt{-\nabla^2 + m^2}\phi(\vec{x})^2 \tag{2.27}$$

$$v = \frac{1}{m}\nabla S = 0 \tag{2.28}$$

$$\delta v = \frac{1}{m}\nabla\delta S = -\int d^3x' \sqrt{-\nabla^2 + m^2}\phi(\vec{x'})\delta^3(\vec{x} - \vec{x'}) = -\sqrt{-\nabla^2 + m^2}\phi(\vec{x}) \tag{2.29}$$

$$v_+ = v + \delta v = -\sqrt{-\nabla^2 + m^2}\phi(\vec{x}) \tag{2.30}$$

For a particle with momentum $\vec{p}$, localized by the wavepacket $f(\vec{p})$, the ground state wavefunction for the corresponding momentum mode is replaced by the wavefunction for the 1st excited state, so the wavefunctional is

$$\Psi[\{\phi_R(\vec{k}),\phi_I(\vec{k})\},t] = \int \frac{d^3p}{(2\pi)^3} f(\vec{p})\psi_1(\phi_R(\vec{p}),t)\psi_1(\phi_I(\vec{p}),t)\prod_{\vec{k}\neq\vec{p}}\psi_0(\phi_R(\vec{k}),t)\psi_0(\phi_I(\vec{k}),t)$$

$$= \int \frac{d^3p}{(2\pi)^3} f(\vec{p})A\phi_R(\vec{p})\phi_I(\vec{p}) \prod_{\vec{k}\text{ independent}} \psi_0(\phi_R(\vec{k}),t)\psi_0(\phi_I(\vec{k}),t)$$

$$= \int \frac{d^3p}{(2\pi)^3} f(\vec{p})A\phi_R(\vec{p})\phi_I(\vec{p}) \exp\left(\int \frac{d^3k}{(2\pi)^3}\left[\frac{-m\omega_{\vec{k}}}{2}(\phi_R(\vec{k})^2+\phi_I(\vec{k})^2) - i\omega_{\vec{k}}t\right]\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} f(\vec{p})A\phi_R(\vec{p})\phi_I(\vec{p}) \exp\left(\frac{-m}{2}\int d^3x\sqrt{-\nabla^2+m^2}\phi(\vec{x})^2\right)\exp\left(-it\int\frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\right)$$

$$= \exp\left(\frac{-m}{2}\int d^3x\sqrt{-\nabla^2+m^2}\phi(\vec{x})^2\right)\exp(-itB)A\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\phi_R(\vec{p})\phi_I(\vec{p}) \quad (2.31)$$

where converting to a position space integral and evaluating $B=\int\frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}$ allows the exponential terms to be pulled out from the d^3p integral in the last line. Here S and v are the same as before. For the other terms,

$$\rho = A^2\exp\left(-m\int d^3x\sqrt{-\nabla^2+m^2}\phi(\vec{x})^2\right)\left(\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\phi_R(\vec{p})\phi_I(\vec{p})\right)^2 \quad (2.32)$$

$$\delta S = -\frac{m}{2}\int d^3x\sqrt{-\nabla^2+m^2}\phi(\vec{x})^2 + \ln\left(\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\phi_R(\vec{p})\phi_I(\vec{p})\right) \quad (2.33)$$

$$\delta v = -\sqrt{-\nabla^2+m^2}\phi(\vec{x}) + \frac{1}{m}\frac{\partial}{\partial\phi(\vec{x})}\ln\left(\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\frac{\phi_{\vec{p}}^2-\phi_{-\vec{p}}^2}{4i}\right)$$

$$= -\sqrt{-\nabla^2+m^2}\phi(\vec{x}) + \frac{1}{m}\left[\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\frac{\phi_{\vec{p}}^2-\phi_{-\vec{p}}^2}{4i}\right]^{-1}\int\frac{d^3p}{(2\pi)^3}\frac{f(\vec{p})}{2i}\left(\phi_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}-\phi_{-\vec{p}}e^{i\vec{p}\cdot\vec{x}}\right)$$

$$= -\sqrt{-\nabla^2+m^2}\phi(\vec{x}) - \frac{\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\phi_I(\vec{p})\sin(\vec{p}\cdot\vec{x})}{m\int\frac{d^3p}{(2\pi)^3}f(\vec{p})\phi_R(\vec{p})\phi_I(\vec{p})} \quad (2.34)$$

Consider now the case of two particles from separate scalar fields ϕ_1 and ϕ_2, with momenta $\vec{p_1}$ and $\vec{p_2}$, and localized by wavepackets $f(\vec{p_1})$ and $g(\vec{p_2})$. The entangled state formed by exchanging the momenta is described by the wavefunctional

$$
\begin{aligned}
\Psi[\{\phi_1(\vec{k})\},\{\phi_2(\vec{k})\},t] &= \frac{1}{\sqrt{2}}\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\left[\prod_{\vec{k}\neq\vec{p_1},\vec{p_2}} \psi_0(\phi_1(\vec{k}),t)\psi_0(\phi_2(\vec{k}),t)\right] \\
&\times [f(\vec{p_1})g(\vec{p_2})\psi_1(\phi_1(\vec{p_1}),t)\psi_1(\phi_2(\vec{p_2}),t) + f(\vec{p_2})g(\vec{p_1})\psi_1(\phi_1(\vec{p_2}),t)\psi_1(\phi_2(\vec{p_1}),t)] \\
&= \exp\left(\frac{-m}{2}\int d^3x\sqrt{-\nabla^2+m^2}\left(\phi_1(\vec{x})^2+\phi_2(\vec{x})^2\right)\right)\exp\left(-2itB\right) \\
&\times \frac{A}{\sqrt{2}}\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\left[F(\vec{p_1},\vec{p_2})\phi_{1R}(\vec{p_1})\phi_{2R}(\vec{p_2}) + F(\vec{p_2},\vec{p_1})\phi_{1R}(\vec{p_2})\phi_{2R}(\vec{p_1})\right] \quad (2.35)
\end{aligned}
$$

where the exponential terms are pulled out in the same way as before and

$$
F(\vec{p_1},\vec{p_2}) = f(\vec{p_1})g(\vec{p_2})\phi_{1I}(\vec{p_1})\phi_{2I}(\vec{p_2}) \quad (2.36)
$$

is used to condense the expression and was chosen based on the final equation of the drift velocity v_+. Calculating v_+ is a long but straightforward extension of the calculation for the single particle case, with the final value being

$$
\begin{aligned}
v_+ = &-\sqrt{-\nabla^2+m^2}(\phi_1(\vec{x})+\phi_2(\vec{x})) - \\
&\frac{\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\left[F(\vec{p_1},\vec{p_2})G(\vec{p_1},\vec{p_2}) + F(\vec{p_2},\vec{p_1})G(\vec{p_2},\vec{p_1})\right]}{m\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\left[F(\vec{p_1},\vec{p_2})\phi_{1R}(\vec{p_1})\phi_{2R}(\vec{p_2}) + F(\vec{p_2},\vec{p_1})\phi_{1R}(\vec{p_2})\phi_{2R}(\vec{p_1})\right]} \quad (2.37)
\end{aligned}
$$

where

$$
G(\vec{p_1},\vec{p_2}) = \phi_{2R}(\vec{p_2})\sin(\vec{p_1}\cdot\vec{x}) + \phi_{1R}(\vec{p_1})\sin(\vec{p_2}\cdot\vec{x}) \quad (2.38)
$$

is used to also used to condense the expression. The stochastic differential equation can then be obtained by substituting Eq. (2.37) into Eq. (2.8) and replacing $q(t)$ with $\Psi[\phi_1(\vec{x}),\phi_2(\vec{x}),t]$.

2.3 The Dirac Field

It should be noted that while Nelson's formalism for stochastic mechanics reproduces the general Schroedinger equation in the nonrelativistic case, in the extension to field theory, the functional Schroedinger equation can only be reproduced in the same way for a Hamiltonian of the form

$$H = -\frac{1}{2m}\nabla^2 + V(\vec{x}). \tag{2.39}$$

The scalar field Hamiltonian has an analogous form, with the field ϕ replacing $\vec{x}$, so the use of the functional Schroedinger equation in the previous derivations was justified, but this is not the case for all fields. In particular, the standard Dirac field ψ has canonical momentum $\pi = i\psi^\dagger$ and Hamiltonian

$$H = \int d^3x \psi^\dagger \gamma^0(-i\gamma^i\partial_i + m)\psi \tag{2.40}$$

which, due to the linearity in derivatives, is clearly not analogous to the nonrelativistic Hamiltonian. While this poses a difficulty for stochastic field theory, a possible method for demonstrating the existence of a stochastic description of the Dirac field is as follows:

Let ψ^α be a 4 component field satisfying the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$. Then each component satisfies the Klein-Gordon equation $(\partial_\mu\partial^\mu + m^2)\psi^\alpha = 0$. The method for deriving the stochastic process for the ground state of the scalar field can then be applied to each component ψ^α, so that they individually each satisfy a stochastic differential equation analogous to Eq. (2.18).

Let ψ be a Dirac spinor and consider the stochastic differential equation

$$d\psi^\alpha = -\sqrt{-\nabla^2 + m^2}\psi^\alpha dt + dw^\alpha. \tag{2.41}$$

This equation is analogous to Eq. (2.18), which corresponds to the scalar field, and can therefore be used to derive the Klein-Gordon equation. We therefore have that

$$-\partial_\mu\partial^\mu\psi^\alpha = m^2\psi^\alpha. \tag{2.42}$$

By casting the Klein-Gordon equation in the form of an operator eigenvalue equation, we can derive the Dirac equation following Dirac's original method, by making the assumption that this operator is the square of some other linear operator. In particular, if we require that

$$-\partial_\mu \partial^\mu = (i\gamma^\mu \partial_\mu)^2 \tag{2.43}$$

we see that the coefficient matrices γ^μ must obey the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.44}$$

which identifies them as the well known gamma matrices. Given that the square of this operator has eigenvalue m^2, we can reasonably assume the eigenvalue is m and recover the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{2.45}$$

It is important to note that although not all solutions to the Klein-Gordon equation will also be solutions to the Dirac equation, the initial assumption that ψ is a Dirac spinor provides the necessary restriction which ensures its components are related in such a way that this move from the Klein-Gordon to Dirac equation is allowed. Unfortunately, the necessity of this assumption means that properties such as spin and fermionic statistics become inherent properties of the field ψ itself and are unrelated to the stochastic process. While it may be possible to derive some other stochastic process which is capable of explaning spin and fermionic statistics, doing so is beyond the scope of this paper. This proof of the existence of stochastic processes from which the Dirac equation can be derived is still relevant as evidence of the compatibility of the stochastic description with fermionic fields though. As a closing observation, Nelson's definition of the acceleration can also be written in terms of the anticommutation of the forward and backward derivates as

$$a(\vec{x}, t) = \frac{1}{2}\{D_{(+)}, D_{(-)}\} q(\vec{x}, t) \tag{2.46}$$

which may be of particular relevance in explaining the anticommutation relations of fermionic fields.

2.4 Perturbations of Stochastic Field Equations

In the absence of a generalized formalism for stochastic field theory, it may be sufficient to know the stochastic description of the ground states for the scalar and Dirac fields. The existence of stochastic processes corresponding to excited and/or interacting states of these fields can then be assumed through application of the Wightman reconstruction theorem, which shows the information of the field theory can be derived from vacuum expectation values of products of operators[7]. A process for relating these expectation values to perturbations of the stochastic differential equations is given in [4] and reviewed briefly here.

Consider two stochastic differential equations

$$dq(t) = v_{(+)}(q(t), t)dt + dw(t) \tag{2.47}$$

$$d\bar{q}(t) = \bar{v}_{(+)}(\bar{q}(t), t)dt + dw(t), \tag{2.48}$$

both defined on the probability space (Q, Σ), the first with measure μ and the second with measure $\bar{\mu}$. Assume the first process $q(t)$ is fully understood and has expectation values

$$E(F(q(t_1), ..., q(t_s))) = \int_Q F(q(t_1), ..., q(t_s))d\mu. \tag{2.49}$$

Assume the expectation values of the second process $\bar{q}(t)$ are

$$E(F(\bar{q}(t_1), ..., \bar{q}(t_s))) = \int_Q F(q(t_1), ..., q(t_s))d\bar{\mu}, \tag{2.50}$$

and that the initial distributions are related by

$$E(F(\bar{q}(0))) = E(F(q(0))\Omega^2(q(0))) \tag{2.51}$$

where $\Omega(x)$ is real and nonnegative and $F(x)$ is smooth. If the measures are related by

$$d\bar{\mu} = \Omega^2(q(0)) \exp\left[\int_0^T \alpha \cdot dw - \frac{1}{2}\int_0^T \alpha^2 dt\right] d\mu, \tag{2.52}$$

then it can be shown that

$$\bar{v}_{(+)}(x,t) = v_{(+)}(x,t) + \alpha(x,t). \tag{2.53}$$

Therefore, the second process can be described through a perturbation of the first process. The Wightman reconstruction theorem[7] then suggests that the existence of all necessary stochastic processes to describe a quantum field theory comes from perturbations of the stochastic process for the vacuum state of the field.

Chapter 3

Diffeomorphism Invariance

If stochastic mechanics is to provide a basis for emergent gravity, it is important that it can be made diffeomorphism invariant. While Lorentz invariance is important for demonstrating consistency with special relativity, this only shows consistency for flat spacetime in which the metric is that of Minkowski space. A theory which can be used in emergent gravity should hold for any arbitrary spacetime, in which the metric is not given but rather derived from the theory through methods outside the scope of this paper. This section will cover the ways in which a stochastic field theory might be made Lorentz invariant, and perhaps diffeomorphism invariant as well.

First, for the ground state of the free scalar field, we can see that the expectation values of the field are Lorentz invariant. In particular, taking the expectation value of ϕ in Eq. (2.18) eliminates the dw term leaving

$$d\langle\phi(\vec{x},t)\rangle = -\sqrt{-\nabla^2 + m^2}\langle\phi(\vec{x},t)\rangle dt. \tag{3.1}$$

Moving the dt to make the left side a time derivative gives something analogous to the square-root of the Klein-Gordon equation, but it is important that the negative in front of the square-root on the right side be preserved after squaring both sides. This can be done by considering the backwards stochastic differential equation, where the velocity is given by $v_{(-)}$. From Eqs. (2.11) and (2.28), we see that in this case $v_{(-)} = -v_{(+)}$. Multiplying both sides by their time-reversed counterpart then gives

$$\frac{\partial^2}{\partial t^2}\langle\phi(\vec{x},t)\rangle = -(-\nabla^2 + m^2)\langle\phi(\vec{x},t)\rangle \tag{3.2}$$

which is equivalent to the Klein-Gordon equation. This shows the expectation values satisfy diffeomorphism invariance, but whether or not the stochastic description of the fields themselves does is a more complicated matter. However, given that the stochastic impulses in this framework occur at discrete points, this also shows that in the intervals between impulses, the stochastic process is Lorentz invariant.

Moving to the case of the single localized particle with momentum $\vec{p}$, the dynamical equation between impulses is

$$\frac{\partial}{\partial t}\Psi(\phi(\vec{x}),t) = -\sqrt{-\nabla^2 + m^2}\phi(\vec{x}) - \frac{\int \frac{d^3p}{(2\pi)^3} f(\vec{p})\phi_I(\vec{p})\sin(\vec{p}\cdot\vec{x})}{m\int \frac{d^3p}{(2\pi)^3} f(\vec{p})\phi_R(\vec{p})\phi_I(\vec{p})} \tag{3.3}$$

Since $f(\vec{p})$ is an arbitrary localizing wavepacket, we can define it to include a factor of $\frac{1}{2E_{\vec{p}}}$ to obtain the Lorentz invariant measure $\int \frac{d^3p}{2E_{\vec{p}}}$. Since ϕ is a scalar field and $f(\vec{p})$ and $\sin(\vec{p}\cdot\vec{x})$ are ordinary functions, applying a Lorentz transformation to the integral terms would be equivalent to a coordinate transformation, so all transforming terms undergo the same covariant transformation. Since the derivatives $\frac{\partial}{\partial x^\mu}$ transform covariantly, the remaining terms in the equation transform covariantly and we can say that the equation as a whole is covariant. In the entangled state, the components of Eq. (2.37) are variations of the same components for the single particle state, and so the dynamical equation between impulses must be covariant in this case as well. In both cases, time and space do not seem to be treated equivalently, and so the form of the equation may not be preserved under arbitrary Lorentz transformations, but demonstrating the covariant transformation of the equations will be important in the next section, which will give a potential solution to this problem. Finally, it should be noted that if the stochastic impulses can be shown to transform covariantly, then the full dynamical description will be covariant, although the exact nature of these impulses is not currently known beyond certain restrictions that must be placed on them.

3.1 Foliations of Space-Time

One concept that may prove useful in demonstrating diffeomorphism invariance in stochastic mechanics is the method of taking a foliation of space-time, which divides it into space-like hypersurfaces referred to as leafes. This approach is motivated by the use of the same technique in Bohmian mechanics to reconcile the nonlocality of the theory with relativity[8][9]. In particular, the leafes of a foliation provides a covariant structure to replace the notion of distinct point in time at which to evaluate the dynamical equations. Notably only the existence, and not the exact structure or governing equations, of a foliation is needed, and while even the existence of a particular partition of space-time may seem nonrelativistic, the mathematical description is consistent with relativity and it has been shown that foliations can be extracted from the wavefunction of a system[8]. This implies that foliations are only nonrelativistic to the same degree as quantum mechanics.

In the context of stochastic mechanics, there has been work by Hakim [10][11] on developing a relativistic description of general stochastic processes which employs a similar approach. Here, the stochastic processes are indexed by 7 dimensional hypersurfaces in the 8 dimensional phase space, which are considered space-like by virtue of having a normal 8-vector which is time-like. While there was not sufficient time to explore this formalism in its entirety and apply it to our present outline of a stochastic field theory, we can still provide the basic argument and possible method for doing so.

We start by assuming the existence of some appropriate foliation whose leaves are spacelike hypersurfaces and which transforms covariantly. We then use the leaves of the foliation as an indexing set on the dynamical equations of the stochastic process, so that these equations are always evaluated using only values which occur in the same leaf. This resolves the question of how to evaluate equations which are inherently nonlocal. In order to maintain the role of the dynamical equations as describing the evolution of a system across the leaves of the foliation, we replace instances of dt with $d\tau$, where τ is the timelike unit normal vector to the given leaf at a given point. This allows movement between leaves while restricting the definition and evaluation of that movement to individual leaves. Since the dynamical equations for particle states, such as Eq. (3.3), were shown to be covariant, and the indexing foliation is covariant, it follows that the indexed dynamical equation will also be covariant.

One potential problem predicted to come from the nonlocality of stochastic mechanics that can also be addressed is what happens if a stochastic impulse causes one particle from an entangled state to be measured first in one frame, but the reverse is true in some other frame. Here we can use the "curved Born rule"[12] which, in short, says that if the wave function for a system has its definition restricted to a Cauchy surface, the probability distribution for measurements made on that Cauchy surface is also restricted to it and defined in the usual manner of $\rho = |\psi|^2$. Here the leaves of the foliation play the role of the Cauchy surfaces, and this result tells us that measurements and probabilities of a system are contextualized by the same foliation used to define the dynamics of the system. If we ensure that the stochastic impulses are also appropriately indexed by the foliation, then this contextualization, along with the covariant nature of the foliation, ensures that the probability of a particular measurement will be the same in all reference frames.

While this approach provides an effective solution to problems of nonlocality, there are potential problems with how to properly extract a foliation. In our model of stochastic mechanics, quantum mechanics is seen as an approximation of the underlying stochastic processes, so the method of extracting a foliation from the wavefunction may no longer work. Taking the existence of a foliation as a given is also a problem, as that conflicts with the relativistic principal that there can be no preferred reference frame, even if such a frame is defined so as to be undetectable. Furthermore, a recent comment[13] on the curved Born rule points out that spacelike hypersurfaces can evolve to no longer be spacelike, and that for plane-wave spacetimes, it has been shown by Penrose[14] that "No spacelike hypersurface exists in the space-time which is adequate for the global specification of Cauchy data." A possible resolution to these problems would be to extract a foliation from the stochastic process in such a way that it can be treated as a self-consistency requirement, rather than a separate structure. This approach is motivated by the apparent success of previous bootstrap models, which operate from a similar approach of self-consistency. Another possibility is the use of clock and ruler fields discussed in [1]. Since these problems were only recently recognized, there was not time to properly explore specific methods of resolving them, but the previously suggest methods indicate that these problems do not seem fundamentally unresolveable.

3.2 The Markov Property

One problem in developing a relativistic stochastic field theory is the existence of theorems[10][15] which were brought up in [16] that show only trivial or deterministic processes satisfy both the Markov property and Lorentz invariance. However, because the stochastic impulses in our model occur at discrete intervals, with fields behaving classically between impulses, the system will retain some 'memory' of its past states within these intervals. Therefore, this description is only approximated by the Markov property, but does not satisfy it at a formal level. Since the relevant theorems make use of this formal definition of the Markov property which holds at all distances, they do not apply to this discretised stochastic model. However, depending on how the stochastic impulses enter into the full stochastic process, it may be that they can be taken separately as a stochastic process which does satisfy the Markov property at a formal level and be subject to these theorems. Since the exact nature of the impulses is not currently known, this is not something that can be presently determined, but these theorems should be kept in mind as something to revisit if the stochastic impulses can in fact be treated as a separate stochastic process added to a classical process.

3.3 Discrete Kicks

In order for the stochastic process to act as a regulator in the theory of emergent gravity, there must be some short distance scale at which stochastic mechanics takes over. Because of this, the random kicks should occur at discrete intervals, rather than continuously. As previously mentioned, this may have the added benefit of avoiding previous problems in making stochastic processes relativistic. However, limiting kicks to discrete intervals has the potential to violate Lorentz invariance, depending on how the points at which kicks occur are distributed. In order to avoid this, we can make use of a particularly useful result from causal set theory[17] which says that if (Ω, Σ, μ) is a probability space for some random process in which a set of points is randomly distributed in n-dimensional Minkowski space, then if the probability measure μ is invariant under composition with Lorentz transformations, meaning $\mu \circ \Lambda = \mu$ for all

$\Lambda \in SO(n-1,1)$, then any set of points distributed in this way is Lorentz invariant. An explicit example of such a process, given by the same paper[17] is a Poisson point process or Poisson 'sprinkling'. A Poisson sprinkling is a locally finite, uniformly random set of points in a manifold for which the probability of finding n points in a region of volume V is

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}, \tag{3.4}$$

where ρ is some fundamental density. Since the probability measure is determined by the volume, and the volume is invariant under Lorentz transformations, this satisfies the necessary properties of the theorem. While having an explicit example of a way in which the stochastic impulses might be distributed is useful, the fact that the actual theorem proved in [17] applies to any random process with a Lorentz invariant probability measure means that the impulses may be distributed in some other manner, such as one defined by some arbitrary foliation, as well.

Chapter 4

Conclusion/Outlook

In this paper we have provided an effective starting point from which a stochastic field theory that gives rise to both quantum mechanics and emergent gravity can be better understood. The basic relationship between stochastic mechanics and quantum mechanics is taken from Nelson[3] while the extension to scalar fields is taken from Guerra and Ruggiero[4][6]. We provide an argument for the existence of stochastic processes corresponding to fermionic fields, as well as an explicit calculation of the stochastic process for an entangled state of 2 particles from separate scalar fields. The question of if the stochastic theory is diffeomorphism invariant is then approached by drawing on developments in Bohmian mechanics[12][8][9] and causal set theory[17]. The use of foliations of space-time in Bohmian mechanics provides a way to index stochastic processes so that they transform covariantly and the question of nonlocality is resolved by restricting evaluations of the equations to each leaf of the foliation. The potential for stochastic mechanics to provide an ultraviolet regulator for emergent gravity requires the stochastic impulses be discrete. Causal set theory provides a Lorentz invariant way to distribute discrete stochastic impulses through a Poisson point process, but the supporting theorem shows that any random process with a Lorentz invariant probability measure will also work.

One of the unresolved questions and potential problems with this theory is understanding the origin of the foliation and stochastic impulses. It is possible that the impulses may arise from deformations in the foliation, or that a foliation can be extracted from the distribution of the impulses. Furthermore, both may be related to the notion of clock and ruler fields discussed in [1]. There may be other explanations, but this seems like the best possibility to explore first. Other topics for future research which build on the results in this paper include a more substantial approach to the stochastic description of fermionic fields, and using the equation derived for entangled scalar fields to investigate the specific role of measurements and the notion of wave function collapse in the stochastic framework. The evidence provided here that stochastic mechanics can be made consistent with relativity makes it promising as a potential resolution to the conflict between quantum mechanics and relativity.

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