Acausality in Higher-Derivative Quantum Mechanics

Senior Thesis

by

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Public Abstract *Introduction, Background and Results:* To understand and predict the way things will move, physicists often use equations of motion. These equations of motion take variables like position, velocity, and acceleration of the object, plus some constants based on properties of the object, like its mass, and put them together in a mathematical form. With these equations of motion, we can figure out how the object will move under given circumstances. This project has manipulated an equation of motion by adding a higher-order time derivative; derivatives are rates of change, so this means that, instead of just having changes to position (velocity) and changes to velocity (acceleration), we have included changes to acceleration, and other higher-order rates of change. When we solve the equation of motion with a higher-derivative, we find that our solutions predict acausal behavior—effects happening before their cause. In a classical system, that would otherwise be governed by Newton's laws, this can look like a ball moving before a bat hits it. The time scale of this acausality, however, would likely be very small, so small that it may not even be in conflict with experimental observation.

The ultimate goal of this project is to evaluate similar acausal behavior in a quantum mechanical system. To do this, we had to rewrite our classical equations in a way that could be carried over to a quantum mechanical theory. In doing this, we defined a new set of variables to describe the system, and this allowed us to write our quantum theory without those higher derivatives. However, this created a new challenge: our system, which originally consisted of one object (our classical ball getting hit by the bat) now was written in terms of two particles: one of these particles looks and behaves as we would expect it to, but the other appears to have a negative energy indicating that we must quantize these particles differently. We believe that if we put these particles in the right system, and quantize them correctly, their presence could lead us to find acausality in this system as well. Demonstrating

this will be the goal of future work.

Intellectual Merit: This project and future work will allow us to investigate the necessity of causality at a microscopic distance scale, thus giving us a better understanding of cause and effect. This work could also potentially contribute to understanding interpretations of quantum field theories that also exhibit acausality.

Broader Impact: This project has fulfilled my degree requirements and will allow me to receive my degree and continue my education to become a high school physics teacher. My degree, coupled with the research experience gained through this project, will allow me to teach countless young physics students and contribute to advancing the next generation of physicists.

Abstract

Some quantum field theories predict divergences that require fine tuning to eliminate. This fine tuning can be reduced or eliminated if higher-derivative terms are present in the theory. We investigate the effects of a higher-order time derivative in a classical system and find that it can produce solutions that exhibit apparent acausality. We then quantize the acausal classical system with the goal of examining the behavior and potential acausality of wave packets in the higher-derivative quantum system.

Chapter 1

Introduction

Quantum field theories predict divergences in integrals describing quantum corrections to various observable quantities. Eliminating the effects of these divergences can sometimes require fine tuning of the theory. The fine-tuning can be reduced or eliminated if the theory has certain kinds of higher-derivative terms. Higher-derivative theories can have useful applications, for example, in producing renormalizable theories of quantum gravity.

In adding higher-order time derivatives, additional boundary conditions must be imposed on the system so that the equations of motion can be solved. However, with the addition of these boundary conditions, these theories predict an apparent violation of causality [1]. An example of this acausality arises in two-particle scattering into two final-state particles. Based on calculations using these higher-derivative quantum field theories, it appears that the new particles are created before the original particles interact.

This project will first add higher-order time derivatives to the Lagrangian of a one-dimensional classical system, apply appropriate boundary conditions, and investigate any acausal behavior. From there, the theory can be quantized by identifying a set of generalized coordinates and conjugate momenta, following the approach of Pais and Uhlenbeck [2], and promoting them to operators. Any apparent acausality arising in the quantum mechanical model will be investigated. For example, we will consider the scattering of wave packets off of a one-dimensional potential. If any acausality arises in higher-derivative, non-relativistic quantum mechanics, it will be readily apparent in the time-dependence of the solutions.

Chapter 2

Developing a Higher-Derivative Quantum Theory

2.1 An Example of Acausality

The solutions to the following equation of motion have acausal solutions [1]

$$\ddot{x} = \lambda \dddot{x} + \delta(t). \tag{2.1}$$

To find this acausality, we work in units where $\dot{x}$ is dimensionless and λ has units of time. Equation (2.1) has the general solution

$$x(t) = ae^{t/\lambda} + bt + c \tag{2.2}$$

in regions where $t \neq 0$, which can be verified by substitution. We can solve Eq. (2.1) by separately looking at times $t < 0$, which we will call region I, and times $t > 0$, which we will call region II. We then have

$$x^I(t) = a^I e^{t/\lambda} + b^I t + c^I \tag{2.3}$$

and

$$x^{II}(t) = a^{II} e^{t/\lambda} + b^{II} t + c^{II}. \tag{2.4}$$

We first assume that the particle is at rest prior to $t = 0$, so the coefficients for region I are all 0, i.e.

$$x^I(t) = 0, \;\; t < 0. \tag{2.5}$$

Continuity of the position means that

$$x^{II}(t=0) = a^{II} + c^{II} = 0. \tag{2.6}$$

Because of delta function, we must look at the discontinuity of the derivatives by integrating over a small range that includes $t = 0$ then taking the limit as that range goes to 0:

$$\int_{-\epsilon}^{\epsilon} (\lambda \dddot{x} - \ddot{x}) dt = -1, \tag{2.7}$$

$$Disc\, \dot{x}|_{x=0} = 0. \tag{2.8}$$

Equation (2.8) must be true because if there was discontinuity in $\dot{x}$,

$$\ddot{x} \propto \delta(t) \tag{2.9}$$

and

$$\dddot{x} \propto \frac{d}{dt}\delta(t) \tag{2.10}$$

which would not solve Eq. (2.1). Therefore, $\dot{x}$ must be continuous, and

$$\dot{x}(0) = \frac{a^{II}}{\lambda} + b^{II} = 0. \tag{2.11}$$

In order to solve Eq. (2.1), there must be a discontinuity in $\ddot{x}$:

$$Disc\, \ddot{x}|_{x=0} = -\frac{1}{\lambda} = \frac{a^{II}}{\lambda^2}, \tag{2.12}$$

Thus $a^{II} = -\lambda$, $b^{II} = 1$, and $c^{II} = \lambda$ and our solution for $t > 0$ is

$$x(t) = -\lambda e^{t/\lambda} + t + \lambda. \tag{2.13}$$

This solution has an exponentially growing term. This "runaway mode" is non-physical, so to eliminate it, we must relax our condition that the all the coefficients are necessarily 0 in region I. If we force a^{II} to be zero, so there is no acceleration after the force acts, and reevaluate, we have

$$x^{I}(t) = a^{I} e^{t/\lambda} + b^{I} t + c^{I} \tag{2.14}$$

and

$$x^{II}(t) = b^{II}t + c^{II}. \tag{2.15}$$

We can apply the condition that at $t << 0$, the particle is at rest at the origin. This sets $b^I = 0$ and $c^I = 0$, so for region I we have

$$x^I(t) = a^I e^{t/\lambda}. \tag{2.16}$$

If we maintain that the function is continuous, or $x^I(0) = x^{II}(0)$, we have

$$x^I(0) = x^{II}(0) = a^I = c^{II}. \tag{2.17}$$

Maintaining that $\dot{x}$ is continuous and $\ddot{x}$ is not,

$$\dot{x}^I(0) = \dot{x}^{II}(0) = \frac{a^I}{\lambda} + b^I = b^{II}, \tag{2.18}$$

and

$$Disc\,\ddot{x}|_{x=0} = -\frac{a^I}{\lambda^2} = -\frac{1}{\lambda}. \tag{2.19}$$

Thus $a^I = \lambda$, $b^{II} = 1$, and $c^{II} = \lambda$, which gives the solution that for $t < 0$,

$$x^I(t) = \lambda e^{t/\lambda} \tag{2.20}$$

and for $t > 0$

$$x^{II}(t) = t + \lambda. \tag{2.21}$$

Equation (2.20) shows that the particle moves before it is acted on by the delta function force $t = 0$. However, this result may not be inconsistent with observations, if the timescale of λ is incredibly small, for instance, on the order of a Planck time.

2.2 Another Example: Starting with a Lagrangian

Our previous example has solutions that display acauasality, but has a major drawback: it is not readily apparent what Lagrangian could give us that equation of motion. In order to investigate acausality in higher-derivative quantum mechanics, we want to start with a higher-derivative Lagrangian, from which we can find a Hamiltonian and construct a quantum theory from the classical one. To do this, we first derive an Euler-Lagrange equation for a Lagrangian with a second-order time derivative. Then, we used that Euler-Lagrange equation to find the equations of motion for a higher-order system with a delta-function potential. From there, we plan to follow the method of Pais and Uhlenbeck [2] to find the Hamiltonian and develop a quantum theory for this system.

2.2.1 Deriving an Euler-Lagrange Equation for a Higher Derivative Lagrangian

Defining a Lagrangian,

$$L \equiv L(x, \dot{x}, \ddot{x}, t), \tag{2.22}$$

we aim find the stationary point of the action,

$$S = \int_{t_1}^{t_2} L(x, \dot{x}, \ddot{x}, t)dt, \tag{2.23}$$

for boundary conditions $x(t_1) = x_i$ and $x(t_2) = x_f$ for fixed positions x_i and x_f. This is analogous to the standard case of a Lagrangian with only first-order time derivatives and the Euler-Lagrange equation.

When the action S, is stationary, small deviations of the action will be 0. These deviations are always 0 at the endpoints, that is $\delta x(t_1) = \delta x(t_2) = 0$. Therefore,

$$\delta S = \int \delta L dt = 0, \tag{2.24}$$

(omitting bounds of integration for convenience). Working through that,

$$\delta S = \int \delta x \frac{\partial L}{\partial x} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}} + \delta \ddot{x} \frac{\partial L}{\partial \ddot{x}} dt \tag{2.25}$$

and integrating the second term by parts once and the third term by parts twice, gives

$$\delta S = \int \delta x \Big(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} \Big) dt. \tag{2.26}$$

Because δS is 0 and δx can be any arbitrary deviation, the terms in parenthesis must always equal 0:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0. \tag{2.27}$$

This is our Euler-Lagrange equation for a higher-derivative Lagrangian.

2.2.2 Delta Potential with a Higher-Order Time Derivative

Now we can use Eq. (2.27) to find the equations of motion of a higher-derivative classical system. We have chosen to use a system with a delta-function potential, similar to the example in Sec 2.1. Since the force is applied at t = 0, intuitively there should be no acceleration of the particle prior to t = 0, which will make any acausality easy to spot.

Defining a Lagrangian,

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\lambda\ddot{x}^2 - \alpha x \delta(t), \tag{2.28}$$

we can use Eq. (2.27) to find an equation of motion:

$$\alpha\delta(t) + m\ddot{x} - \lambda \ddddot{x} = 0. \tag{2.29}$$

At times $t \neq 0$, we can solve this to find

$$x(t) = a_+ e^{\omega t} + a_- e^{-\omega t} + bt + c \tag{2.30}$$

where $a, b,$ and c are constants and $\omega = \sqrt{\frac{m}{\lambda}}$, where we have assumed that $\frac{m}{\lambda} > 0$. To solve for these coefficients, we know that $x(t)$ and $\dot{x}(t)$ are continuous, and thus must match at $t = 0$. We can evaluate the function at $t < 0$, which we will call region I, and at $t > 0$, which we call region II, then match these solutions at $t = 0$.

To prevent nonphysical runaway solutions, a_-^I and a_+^{II} must both be 0, preventing the solution from blowing up at $\pm\infty$. Let us consider a solution where $a_-^{II} = 0$, so that there is no acceleration after $t = 0$, when no force is applied, a physically reasonable choice. With this condition we will have a constant velocity after $t = 0$. We can also say that at $t \to -\infty$, our particle is at rest at the origin, thus $b^I = 0$ and $c^I = 0$. This leaves us with the following two equations:

$$x^I(t) = a_+^I e^{\omega t} \tag{2.31}$$

and

$$x^{II}(t) = b^{II} t + c^{II}. \tag{2.32}$$

Following a similar approach to our previous example, we assume continuity of the function and its first derivative,

$$x^I(0) = x^{II}(0) = a_+^I = c^{II}, \tag{2.33}$$

$$\dot{x}^I(0) = \dot{x}^{II}(0) = a_+^I \omega = b^{II}. \tag{2.34}$$

Discontinuity of the third derivative gives

$$Disc\, \dddot{x}^I|_{x=0} = \frac{\alpha}{\lambda} = -a_+^I \omega^3. \tag{2.35}$$

This means that $a_+^I = c^{II} = -\frac{\alpha}{\omega^3 \lambda}$ and $b = -\frac{\alpha}{\omega^2 \lambda}$. Therefore, for $t < 0$

$$x(t) = -\frac{\alpha}{\omega^3 \lambda} e^{\omega t} \tag{2.36}$$

and for $t > 0$

$$x(t) = -\frac{\alpha}{\omega^2 \lambda} t - \frac{\alpha}{\omega^3 \lambda}. \tag{2.37}$$

This shows that even though no force acts until $t = 0$, the particle accelerates before $t = 0$. Hence, this solution displays acausal behavior.

2.3 Deriving a Quantum Mechanical Theory from a Higher-Derivative System

Following the method of Pais and Uhlenbeck [2], we plan to rewrite our Lagrangian from Sec 2.2.2 in terms of generalized coordinates. Their method will allow us to write a Hamiltonian that does not contain higher-derivative terms.

For a one-dimensional, higher-derivative system of finite order,we can rewrite the Lagrangian as

$$L = -xF(D)x, \tag{2.38}$$

where the equation of motion is

$$F(D)x = 0, \tag{2.39}$$

and

$$D = \frac{d}{dt}. \tag{2.40}$$

Using Eq. (2.40), we can rewrite our Lagrangian,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\lambda\ddot{x}^2 \tag{2.41}$$

as

$$L = -\frac{1}{2}mxD^2x - \frac{1}{2}\lambda xD^4x. \tag{2.42}$$

We have changed our sign convention for λ as a matter of convenience, and temporarily ignored the potential. This allows us to define $F(D)$ as

$$F(D) = \frac{1}{2}mD^2 + \frac{1}{2}\lambda D^4. \tag{2.43}$$

To write a Hamiltonian from this Lagrangian we can use the method defined by Ostragradski for a polynomial F [2]. We define "position" coordinates, Q_i:

$$Q_1 = \frac{D^2x}{\omega^2} \tag{2.44}$$

$$Q_2 = (1 + \frac{D^2}{\omega^2})x \tag{2.45}$$

where $\omega^2 = \frac{m}{\lambda}$.

As Pais and Uhlenbeck, we can then write a new Lagrangian,

$$\bar{L} = -\eta_1 Q_1 (D^2 + \omega^2) Q_1 - \eta_2 Q_2 D^2 Q_2. \tag{2.46}$$

This Lagrangian will allow us to easily write down the Hamiltonian. Therefore, if we find the values of η_1 and η_2 that give us a Lagrangian equivalent to Eq. (2.42), we can construct a Hamiltonian written without higher derivatives, from our higher-derivative Lagrangian.

To find η_1 and η_2, we first plug in the values of Q_1 and Q_2 into $\bar{L}$ and simplify:

$$\bar{L} = -\eta_1 \frac{D^2 x}{\omega^2}(D^2 + \omega^2)\frac{D^2 x}{\omega^2} - \eta_2 (1 + \frac{D^2}{\omega^2}) x D^2 (1 + \frac{D^2}{\omega^2}) x \tag{2.47}$$

$$= x(-\eta_1 \frac{D^4}{\omega^2}(1 + \frac{D^2}{\omega^2}) - \eta_2 D^2 (1 + \frac{D^2}{\omega^2})^2) x. \tag{2.48}$$

Then, rearranging L,

$$L = x(-\frac{1}{2} m D^2 - \frac{1}{2} \lambda D^4) x. \tag{2.49}$$

By matching powers of D, we see that L has no D^6 term, therefore the D^6 terms of $\bar{L}$ must cancel, thus

$$\eta_1 = -\eta_2. \tag{2.50}$$

Then, if we match D^2 terms of L and $\bar{L}$ (as evaluating the D^4 terms will give us the same result, just a little less simply),

$$-\eta_2 D^2 = -\frac{1}{2} m D^2 \tag{2.51}$$

meaning

$$\eta_2 = \frac{m}{2} \tag{2.52}$$

and

$$\eta_1 = -\frac{m}{2}. \tag{2.53}$$

This gives a Lagrangian

$$L = \frac{m}{2}Q_1(D^2 + \omega^2)Q_1 - \frac{m}{2}Q_2 D^2 Q_2. \tag{2.54}$$

Defining the conjugate momenta, $P_i = \frac{dL}{d\dot{Q}_i}$ as usual, one finds $P_1 = -m\dot{Q}_1$ and $P_2 = m\dot{Q}_2$. This leads to the Hamiltonian

$$H = -\frac{1}{2m}P_1^2 - \frac{1}{2}\omega^2 Q_1^2 + \frac{1}{2m}P_2^2. \tag{2.55}$$

Fortunately, we have arrived at a Hamiltonian written with no higher-derivative terms. However, it now reveals that our system has two particles! One particle with position Q_2 and momentum P_2 looks like a sensible, free particle. The other particle similarly has position Q_1 and momentum P_1 and looks like particle in a harmonic oscillator potential. However, the sign of these terms is opposite to the case of a "real" particle in a similar potential.

2.4 A Plan to Evaluate the Hamiltonian

Now that we have a Hamiltonian, we need to understand how to modify the Hilbert space so that the energy is not unbounded from below. In Lee-Wick theories [3], this is done by changing the Hilbert space metric to allow states with negative norm. If we succeed in this, then we would like to define some potential, then evaluate the system's time evolution, hopefully allowing us to observe acausal behavior in scattering.

The coordinates Q_1 and Q_2 represent particle positions in one dimension. We could constrain our unusual particle, with position Q_1, by creating a infinite potential well in Q_1 space that forbids this particle from contributing beyond some small region. This would be analogous to Lee-Wick theories, where the Lee-Wick particles are excluded from scattering states that can reach spatial infinity. We could then create a wave packet out of our other particle, with position Q_2, and have it interact with some potential, $V(Q_1, Q_2)$. If we used a time-dependent delta function, we could potentially recreate our classical problem from Sec. 2.2.2. and observe an acausal solution.

Chapter 3

Conclusion

If higher-derivatives are included in a classical Lagrangian, we can use an Euler-Lagrange Equation for higher-derivative Lagrangians to find an equation of motion. From that equation of motion, it is possible to arrive at solutions that exhibit acausality. Evaluating the case of a delta potential with a higher-order time derivative, we have shown solutions that show a particle moving before the delta force is applied, thus displaying acausal behavior. Following the method of Pais and Uhlenbeck [2], we were able to rewrite our higher-derivative Lagrangian in such a way that the Hamiltonian of the system does not have higher derivatives, allowing us to make a quantum theory. In the future, we will consider how to modify the Hilbert space of the theory to avoid negative energies and apply a potential to this now quantum-mechanical higher-derivative system to investigate if acausality, analogous to our classical example, can be found. This could potentially provide a link between the acausality found in classical example of higher-derivative theories and that found in the fine-tuning of quantum field theories with higher-derivatives.

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