Modeling Landau damping using the lattice Boltzmann method

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Abstract

In its simplest form, Landau damping describes a phenomena where an energy exchange occurs between an electrostatic wave and plasma particles whose velocities are close to the phase velocity of the wave. For a Maxwellian plasma this results in wave damping, despite the fact that the Vlasov equation is time-reversible. While this kinetic model is fully functional and experimentally verified, the equation requires many degrees of freedom and therefore comes with a high computational cost. Fluid equations provide a convenient reduction in the description of a problem, allowing for fewer variables and increased computability. While there exists fluid models that can recover the effects of Landau damping, this project seeks to use the lattice Boltzmann method to achieve comparable results. Unlike nonlocal computational fluid dynamic codes, the lattice Boltzmann method is highly parallelizable, allowing for increased simplicity of computational models. The goal of this research project is to verify the ability of the lattice Boltzmann formulation to recover the effects of Landau damping.

1 Introduction

1.1 Landau damping

High frequency Landau damping describes the particle-wave interaction between rapidly oscillating charged particles in a plasma and electrostatic waves, also known as Langmuir waves. It is interesting to note that in a vacuum only electromagnetic waves can propagate, but a plasma medium allows for the propagation of electrostatic waves. When considering high frequency oscillations in the electron plasma frequency range, one can ignore the dynamics of the massive ions which cannot respond to the high frequency. Subsequently, the model only needs to consider the electron dynamics. Near equilibrium, electron velocities can be modeled by a Maxwellian distribution, and this will lead to the damping of the electric field. For more general plasma distributions the phase velocity of the interacting electrostatic wave determines whether more net energy is transferred from the particles to the wave (leading to an instability) or from the wave to the electrons (leading to wave damping). It is important to note that only the particles with velocities approximately equal to the phase velocity of the wave will have such strong interactions.

The physical interpretation can be a useful tool for understanding the mechanics of Landau damping. If a surfer is paddling slower than an oncoming wave, energy from the wave will transfer to the surfer and increase his velocity. Similarly, a surfer paddling slightly faster than the wave will catch up to the next crest and begin paddling uphill, expending more energy by transferring it to the wave.

1.2 Vlasov-Poisson equation

Vlasovs equation can be used to model a collisionless plasma. Describing the time evolution of the electron distribution, this equation self-consistently accounts for the kinetic effects caused by the electric field. Landau damping does not require a magnetic field or magnetic perturbations. Obviously in magnetic fusion devices there are strong magnetic fields, but to understand the basics of Landau damping one can work strictly with an electrostatic plasma, ignoring all magnetic field effects. Defining the electron distribution function as $f = f(\vec{x}, \vec{u}, t)$, we can express the following relationship:

$$\frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f + \frac{q\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{u}} = 0 \tag{1}$$

where $\vec{E}$ is the electric field and $\frac{q}{m}$ is the charge-to-mass ratio of an electron. Since we are interested in electrostatic waves, we need only consider Poisson's equation for the electric field (see Appendix for details). This system of equations can be used to determine the rate of damping of the electric field induced by the energy transfer between an electrostatic wave and the electrons with corresponding velocities near the phase velocity of the wave. For a single-maximum distribution, there are a larger number of electrons whose velocities are less than the phase velocity of the electrostatic wave. The wave has a net energy loss and is therefore damped.

1.3 Kinetic vs. fluid model

The Vlasov equation is a kinetic model that takes into account both the position and the velocity of the electron distribution in a plasma. Consider using Eq. (1) to model in three dimensions: the model requires three spatial variables to determine the position, three velocity components to determine the velocity, and one time variable. In total, the 3D kinetic model requires 6+1 independent variables which in turn introduces computational limitations such as large memory demands and advanced computing techniques such as cluster processing.

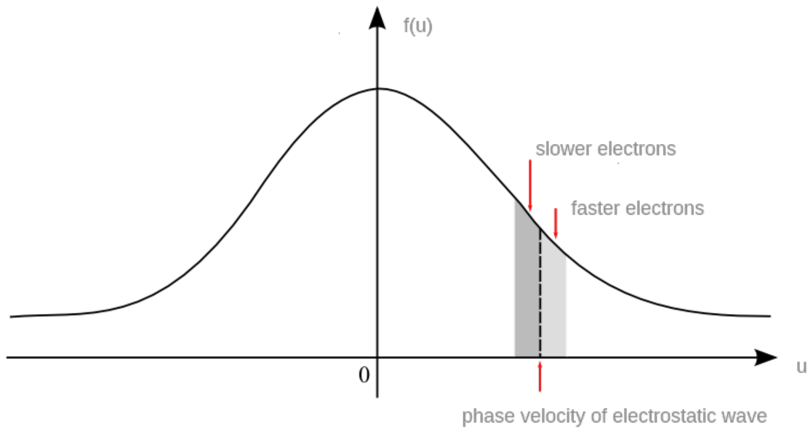

Figure 1: This image shows a distribution of electron velocities. The electrons whose velocities are approximately the same as the phase velocity of the electrostatic wave interact strongly with the wave. It can be seen that there are more electrons with a velocity slower than the phase velocity of the electrostatic wave, causing a net energy transfer from the electrostatic wave to the slower electrons, therefore inducing a damping effect.

As a result, researchers have successfully attempted to take a macroscopic approach to modeling the distribution of electrons in a plasma. At first sight this seems an impossible task since at the macroscopic level one would assume that detailed local kinetic effects like wave-particle interactions would be removed by integrating over all kinetic velocity effects. However clever implementation of nonlocal spatial structures in the fluid equations can recover these effects.[1] Rather than using a kinetic model that tracks individual particle motion, one can create a fluid model that tracks macroscopic properties including density (ρ), momentum ($\rho\vec{v}$) and pressure (P). These properties are derived by taking the moments of the distribution:

$$\rho = \int f dv,\ \rho\vec{v} = \int f\vec{v} dv,\ P = m\int f dv(u-\vec{v})^2 \tag{2}$$

where $\rho = \rho(\vec{x}, t)$. Integrating the Vlasov equation in terms of the above-defined moments, we obtain the following relationships:

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \tag{3}$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla(\rho\vec{v}\vec{v}) = \frac{q\vec{E}\rho}{m} - \nabla P \tag{4}$$

These equations show that taking progressively higher moments continuously introduce new undefined variables, and therefore create an unclosed hierarchy which requires a closure approximation. Fick's law of diffusion relates the momentum of the fluid to the spatial change in the density.

$$\Gamma \equiv \rho\vec{v} \approx -D\nabla\rho \tag{5}$$

where $\Gamma = \Gamma(\vec{x}, t)$ and D is a diffusion coefficient. In order to easily take advantage of this relationship we must express Eq. 6 in Fourier space

$$\Gamma_k = -D_k ik\rho_k \tag{6}$$

If we define the Fourier transformed diffusion coefficient to be $D_k = \sqrt{\frac{2}{\pi}}\frac{v_{th}}{|k|}$, we can re-transform the relation back into Cartesian space such that

$$\Gamma = -\frac{\sqrt{2}v_{th}}{\pi^{\frac{3}{2}}}\int_0^\infty dx' \frac{\rho(\vec{x}+x') - \rho(\vec{x}-x')}{x'} \tag{7}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\Gamma \tag{8}$$

If we again consider modeling the Vlasov equation in three dimensions, we can see that the fluid model requires three spatial variables and one time variable, decreasing the number of required free variables from seven to four. Contrary to the kinetic model in which the distribution function is defined in terms of velocity, position and time, the fluid model has fewer free variables and therefore lower computational demands.

1.4 Lattice Boltzmann method

The lattice Boltzmann method is used in computational fluid dynamics to simulate Newtonian fluids through a simple sequence of collision and streaming processes. This method establishes a mesh of nodes that represent the particles of a fluid which exchange information on neighboring particles motion. Since only neighboring nodes exchange information, this method is highly parallelizable. The lattice Boltzmann method, when solved on a minimal velocity lattice, reduces to fluid equations.[2] This property makes the method an excellent candidate for developing a fluid model. It should be stressed that while the lattice Boltzmann approach in no way tries to mimic the Vlasov equation, it attempts to mimic the fluid equations under consideration. The lattice Boltzmann method makes use of the following relation:

$$f_i(\vec{x} + \vec{c_i}\delta t, t + \delta t) = f_i(\vec{x}, t) + \frac{1}{\tau}(f_i^{eq} - f_i) \tag{9}$$

where c_i is a velocity vector, f_i^{eq} is an equilibrium distribution function and τ

is a relaxation time that controls the rate at which one approaches the equilibrium distribution f_i^{eq}. One of the more useful aspects of the lattice-Boltzmann method is the way it relates different macroscopic quantities such as density and pressure to the equilibrium state of the distribution function. Using generalized velocities c_i defined by the equilibrium distribution function, the method allows for the simple computation of the particular movement in a fluid.

$$\rho = \sum_i f_i^{eq} \tag{10}$$

$$\rho\vec{v} = \sum_i f_i^{eq} c_i \tag{11}$$

Because the lattice Boltzmann method relates the macroscopic properties for density and momentum to an equilibrium distribution function, it is a workable approach to creating a computationally elegant simulation for Landau damping. This project seeks the use the formulation of the lattice Boltzmann method to recover the kinetic effects of Landau damping that would normally be absent from straightforward fluid models.

It is worth noting that while the lattice Boltzmann method operates by modeling the streaming and collision processes between particles, we will be implementing this method in a collisionless Vlasov plasma.

2 Methods

To model Landau damping using a lattice-Boltzmann formulation, we begin by limiting the scope of the problem by developing a one dimensional model. Likewise, we consider the standard case of no equilibrium electric field, that is $E_0 = 0$. Given this condition, we can look at linear solutions to the Vlasov equation in 1D in the following form:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0 \tag{12}$$

In order to redefine this relationship as a fluid equation, we use the fluid moments of the density and moments to rewrite the equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\Gamma)}{\partial x} = 0 \tag{13}$$

In order to relate the density to Γ, the closure solution, we implement the lattice Boltzmann method and define a mesh where the nodes represent a subset of the particles in a plasma, modeling Eq. 7 and 8.

2.1 D1Q3

An initial approach is to use a three bit model[2] to see whether or not it has the ability to effectively describe Landau damping.

Figure 2: The D1Q3 model allows for the simulated particles to move either left, right or not at all.

In order to implement the lattice Boltzmann method, we must first define a equilibrium distribution function that not only matches the geometry of the model but also satisfies the conditions detailed in Eq. 10 and 11. One choice is:

$$f_{eq} = \frac{1}{3}\rho + \frac{1}{2}c_i\Gamma \tag{14}$$

where $i = -1, 0, 1$. With the particular motion of the plasma generalized into three distinct options, we can now define the streaming and collision processes that drive the time evolution of the model.

$$f_i^{collide}(x,t) = f_i(x,t) + \frac{1}{\tau}(f_i^{eq} - f_i) \tag{15}$$

$$f_i^{steam}(x + c_i\delta t, t + \delta t) = f_i^{collide}(x,t) \tag{16}$$

Equations 13-16 are the basis of the fluid model by which electrostatic wave decay can be observed. In order to observe these predicted damping effects, we use a perturbation of a sinusoidal density function that has a range of 0 to 2π. A program simulates the streaming and collision processes and outputs 20 points along the time evolution, generating an increasingly damped waveform at each time step. We generated various outputs for this model, using different values for the initial density perturbation $\partial\rho$ as well as for the relaxation time τ.

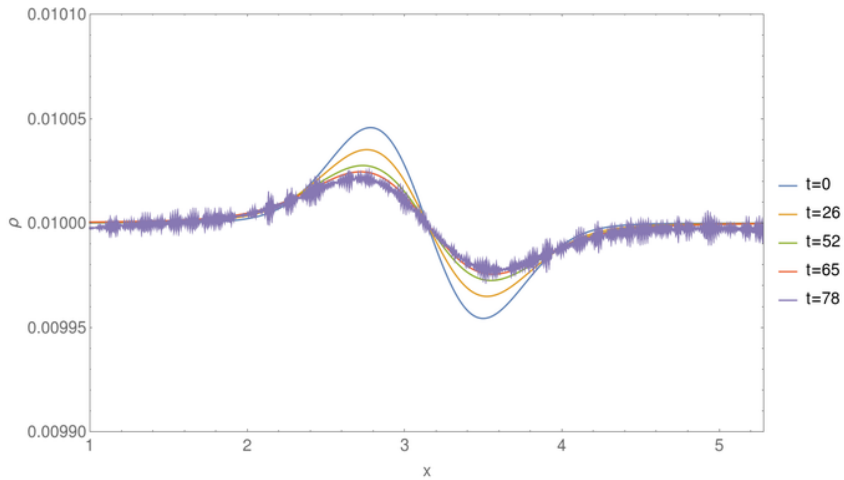

Figure 3: $\frac{\partial \rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.58$

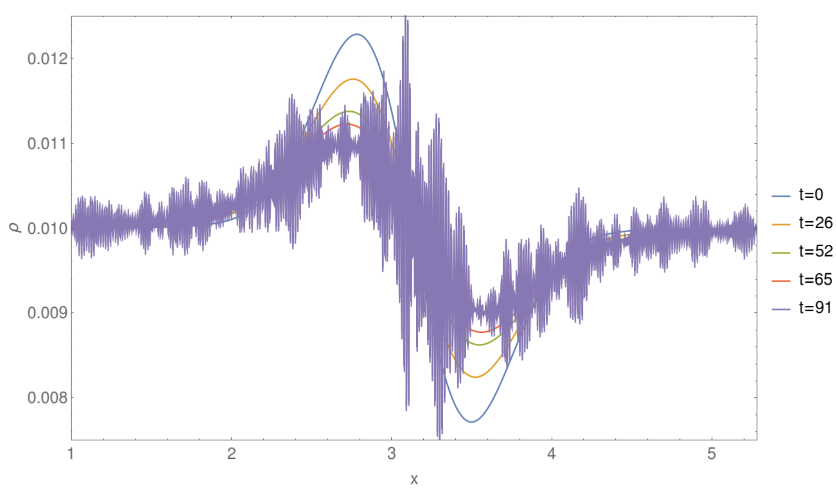

Figure 4: $\frac{\partial \rho}{\rho} = 20\%$ perturbation and $\tau = 0.58$

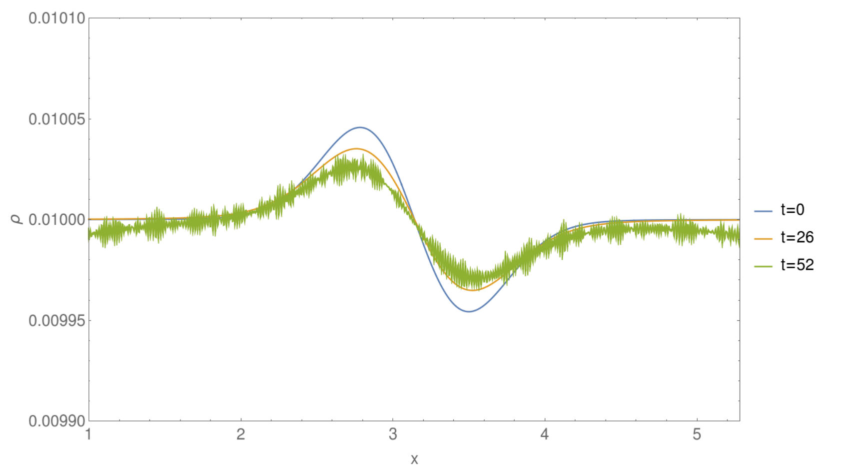

Figure 5: $\frac{\partial \rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.525$

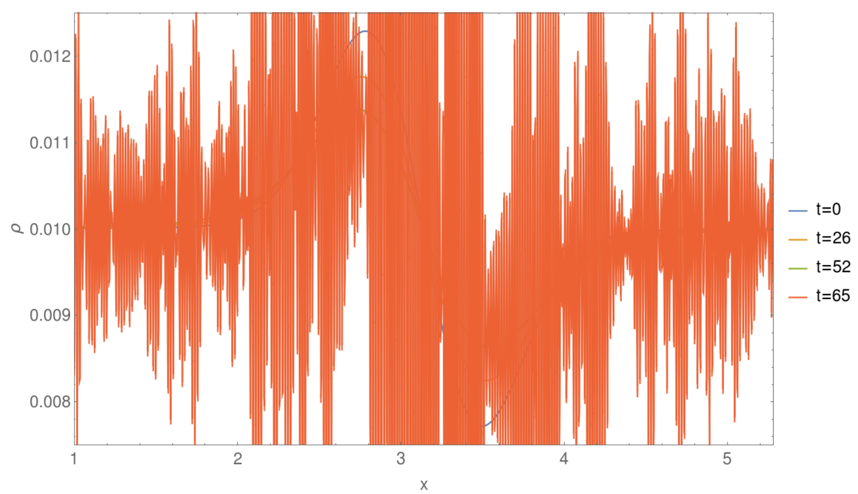

Figure 6: $\frac{\partial \rho}{\rho} = 20\%$ perturbation and $\tau = 0.525$

As seen above, the perturbed density function decays to zero as time increases. This suggests that the model has the ability to recover the Landau damping effects observed in plasma. However, it is clearly seen that the higher time steps are susceptible to numerical instabilities, which distort the results. An increase in the initial density perturbation resulted in increased numerical instability at higher time steps; in the same way bringing τ closer to $\frac{1}{2}$ also drastically increased the models' instability. Without the ability to identify a global maximum at higher time steps, it is impossible to determine if this formulation exhibits exponential decay. This begs the question of whether or not these instabilities can be removed using a more complex model.

It is relevant to mention that while the scope of the research is to discover computationally simple methods for modeling Landau damping, we decided to create a more complex model to correct for said instabilities. A more complex model sacrifices a minimization in computing time for increased numerical stability.

2.2 D1Q5

In order to correct for the numerical instabilities that occur in the D1Q3 model, we can implement a model with increased discretion for the generalized velocity vectors. While the D1Q3 model limits the range of particle velocities to ± 1 in lattice units, this new model includes additional velocities ± 2.

Figure 7: The D1Q5 model allows for the simulated particles the freedom to move left and right with two different magnitudes, as well as the option to not move at all.

While the D1Q3 and D1Q5 models appear to have the same geometry, the addition of two bits allows enough freedom to eliminate numerical constraints at end points. With stable end points, the model is able to correct for undesired oscillations. This model requires a new expression for the equilibrium distribution function:

$$f_{eq} = \frac{1}{5}\rho + \frac{1}{10}c_i\Gamma \tag{17}$$

where $i = -2, -1, 0, 1, 2$. Using this adjusted function allows increased numerical flexibility for the model; having stable time evolution enables the determination of a definite global maximum at every time step, allowing for the precise observation of the Landau decay.

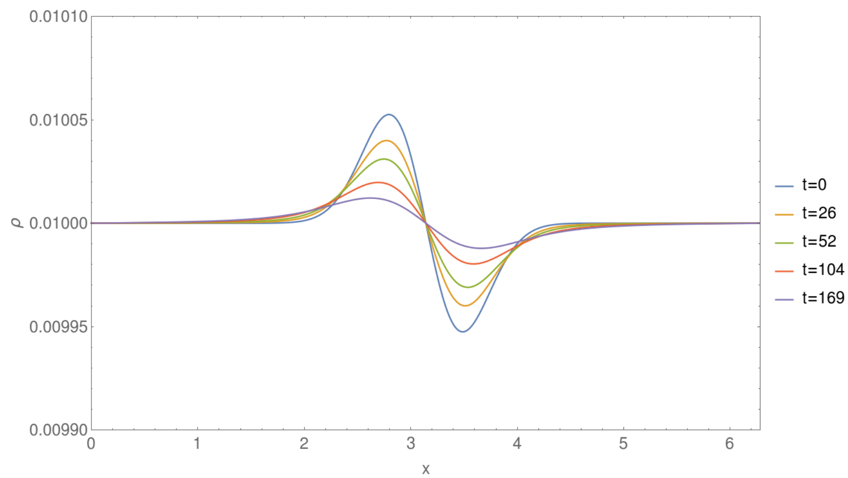

Figure 8: $\frac{\partial \rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.58$

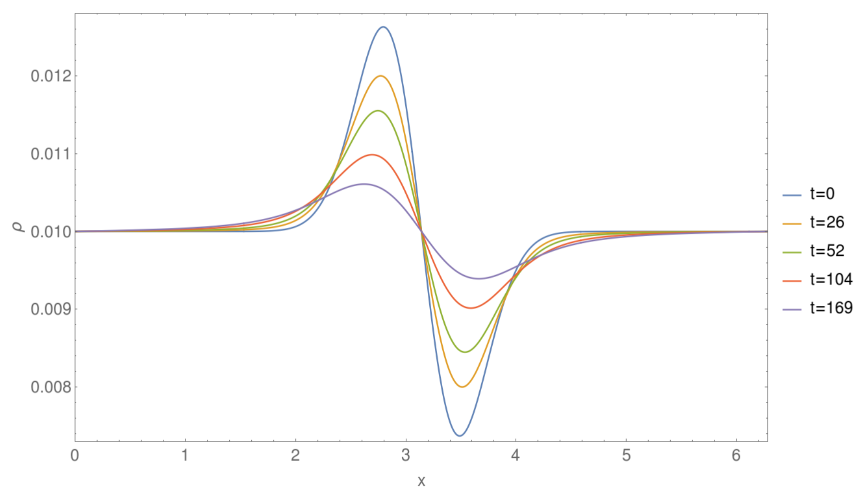

Figure 9: $\frac{\partial \rho}{\rho} = 20\%$ perturbation and $\tau = 0.58$

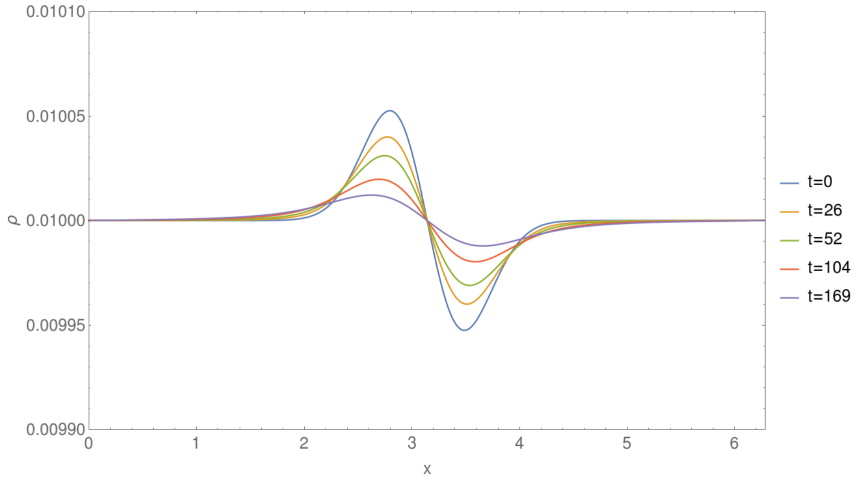

Figure 10: $\frac{\partial\rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.525$

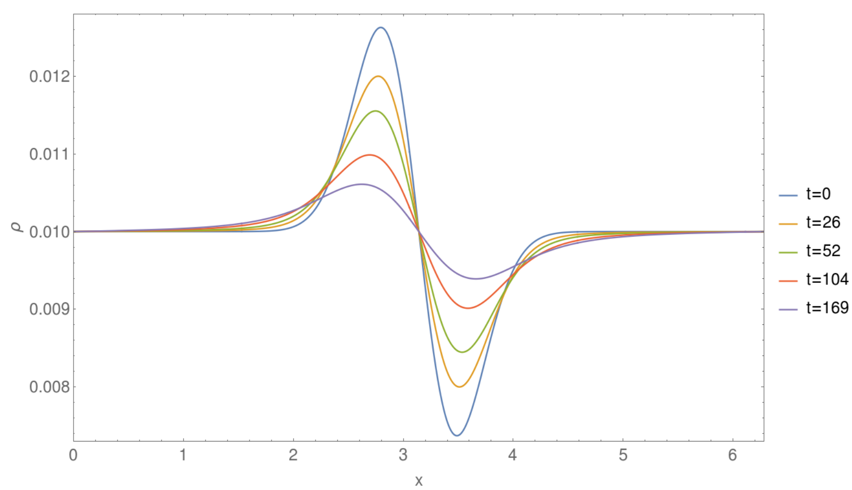

Figure 11: $\frac{\partial\rho}{\rho} = 20\%$ perturbation and $\tau = 0.525$

As seen above, the five bit model effectively removes the numerical instabilities exhibited by the three bit model. Additionally, adjustments to the initial density perturbation showed to have no effect on the numerical stability of the model. Most interestingly, up to a certain point adjustments to the relaxation time also had no effect on the numerical stability of the model. However, as expected, there is a breakdown in the five bit model as $\tau \to 0$; numerical instabilities started becoming apparent at $\tau = 0.515$.

Due to the fact that we were able to correct the numerical instabilities by increasing the complexity of the model, we are able to more accurate determine the global maximum for each time step. This will allow us to determine whether or not the results exhibit exponential Landau damping,

3 Results

The formulation of the Vlasov equation in the context of the lattice-Boltzmann method showed to effectively recover Landau damping. While numerical instabilities proved to be an issue with the computationally simplest model, a minor increase in the complexity of the equilibrium distribution function corrected this drawback. With a numerically smooth model, we can observe the damping over time by identifying the global maximum for each time step and subsequently plotting these values at each time step.

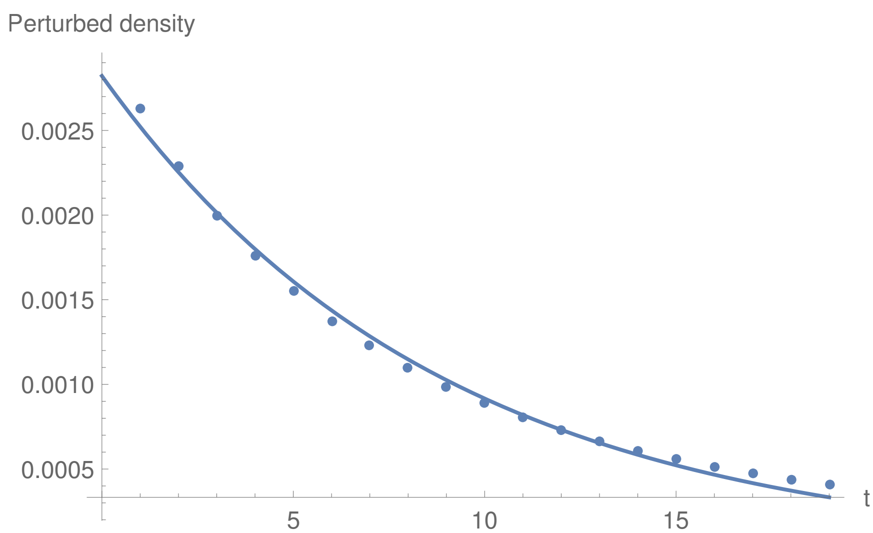

Figure 12: This plot shows the exponential decay of the maximum value at each lattice Boltzmann time step.

Pictured below are the ten latest time steps for the five bit model with initial conditions corresponding to Fig. 8-11. These final time steps were fitted to a log-linear plot where the slope is the damping constant.

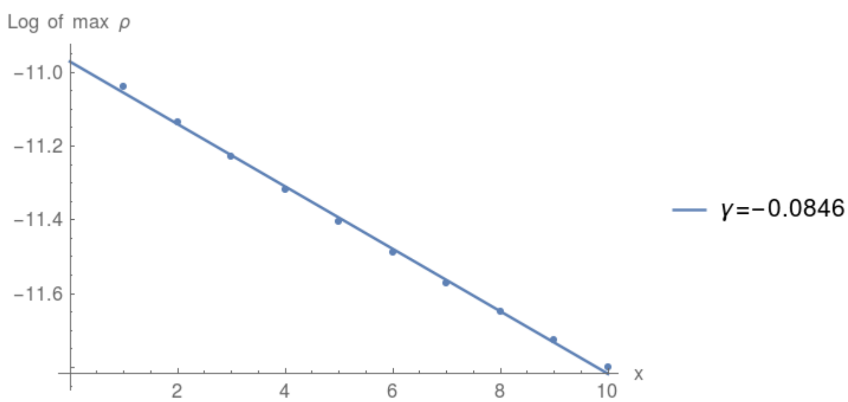

Figure 13: $\frac{\partial\rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.58$

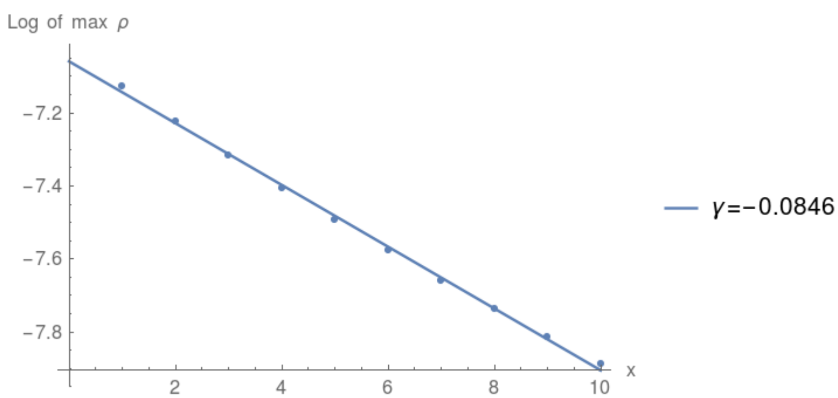

Figure 14: $\frac{\partial\rho}{\rho} = 20\%$ perturbation and $\tau = 0.58$

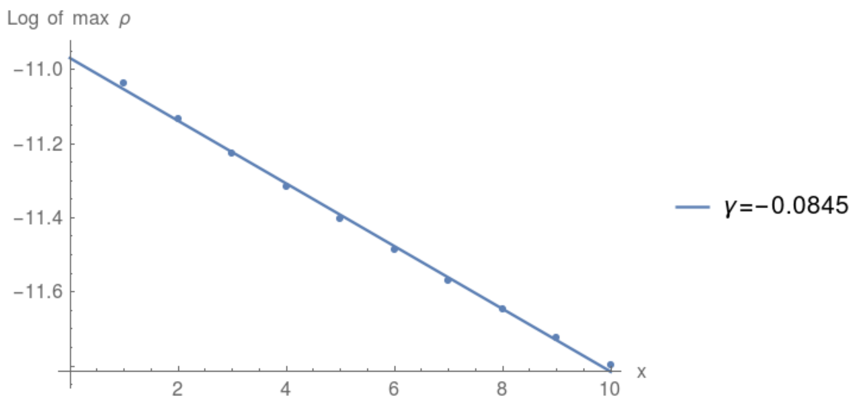

Figure 15: $\frac{\partial\rho}{\rho} = 0.5\%$ perturbation and $\tau = 0.525$

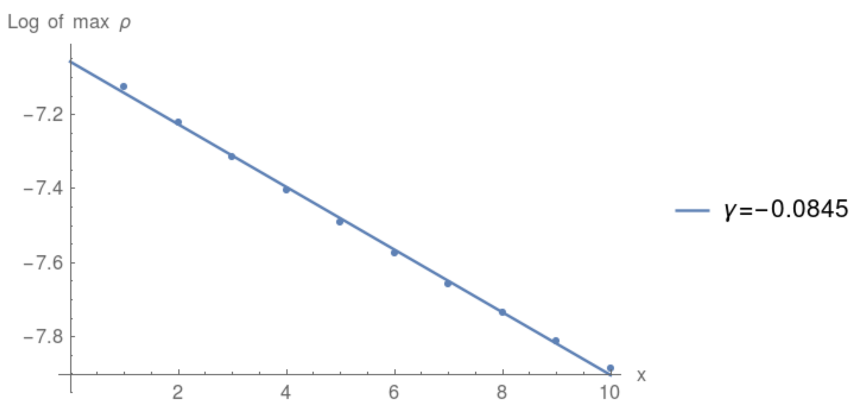

Figure 16: $\frac{\partial\rho}{\rho} = 20\%$ perturbation and $\tau = 0.525$

As can be seen, the five bit model returns the same damping constant regardless of the initial density perturbation or the relaxation time. This suggests that the five bit model is sufficiently numerically stable to not only return the damping effects but to consistently return the same damping constant given different parameters.

4 Discussion

This project sought to recover the effects of Landau damping using a fluid model and solved by a lattice Boltzmann technique. One of the shortcomings of using a fluid model to represent a kinetic phenomena is the absence of information about particular position. Formulating the Vlasov equation in terms of the lattice Boltzmann method attempts to solve the nonlocal fluid equations.

It has been exhaustively mentioned that Landau damping occurs in collisionless plasma while the lattice Boltzmann method uses steaming and collision processes. It is therefore worth acknowledging that the fluid model formulation introduces a non-native component of motion to a system that attempts to describe a kinetic model with fewer spatial components. The loss of information from the kinetic model seems to be compensated for in the collision process of lattice Boltzmann method as implemented in the fluid model.

While the five bit model was able to produce results with high precision, it is important to include a disclaimer about their accuracy. Most expansions for the model were taken no higher than the first order, so the accuracy of the exact numerical value of the results is difficult to determine. However the model does indicate that to leading order, the damping effect can be observed.

This prompts a discussion about the complexity and computability of the model. Using higher moments as the basis for the fluid formulation would undoubtedly increase the accuracy of the result. This would come at the expense of increased computing time, larger processing demands, and increased limitations in parallelization.

5 Conclusion

Using the lattice Boltzmann method, we formulated a fluid model that recovered the effects of Landau damping. We used a streaming and collision process to model the energy dissipation of electrons in a collisionless plasma. This formulation used fluid density as a proxy for the electron distribution of a plasma and a perturbation of the density to model the dissipative effects. We were able to observe these effects and fit the observed data to an exponential decay function. Moreover we were able to demonstrate the ability of the lattice Boltzmann method to simulate kinetic models, offering a workable approach to developing models that are highly parallelizable and computationally simple.

6 Acknowledgement

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7 References

1 G.W. Hammett, W. Dorland, F.W. Perkins, Phys. Fluids B 4 (1992) 2052

2 A. Macnab, G. Vahala, L. Vahala, P. Pavlo, M. Soe, Czech. J. Phys. 362 (2006) 49

A Calculations

Consider the nonlinear electron Vlasov-Poisson eqaution in 1D:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial u} = 0 \tag{18}$$

$$\frac{\partial E}{\partial x} = 4\pi q n_0[1 - \int f du] \tag{19}$$

We need to linearize these equations assuming $E_{ext} = 0$. The first order perturbation of f and E, for $\epsilon << 1$:

$$f = f(u) + \epsilon f^{(1)}(x,u,t) \tag{20}$$

$$E = 0 + \epsilon E^{(1)}(x,t) \tag{21}$$

The linearized equations are:

$$\frac{\partial f^{(1)}}{\partial t} + u\frac{\partial f^{(1)}}{\partial x} + \frac{qE^{(1)}}{m}\frac{\partial f^{(0)}}{\partial u} = 0 \tag{22}$$

$$\frac{\partial E^{(1)}}{\partial x} = -4\pi q n_0 \int du f^{(1)}(x,u,t) \tag{23}$$

Moving to a Lagrangian representation following particle orbit:

$$\frac{\partial f^{(1)}}{\partial t} + u\frac{\partial f^{(1)}}{\partial x} = \frac{\partial}{\partial \tau} f^{(1)}(x(\tau), u(\tau), \tau) \tag{24}$$

$$\text{with } u(t) = u \rightarrow u(\tau) = u = constant \tag{25}$$

$$\text{and } x(t) = x \rightarrow x(\tau) = x + u(\tau - t). \tag{26}$$

Thus:

$$f^{(1)}(x,u,t) - f^{(1)}(x-ut,u,0) = \frac{-q}{m}\frac{\partial f^{(0)}}{\partial u}\int_0^t d\tau E^{(1)}(x(\tau),\tau) \tag{27}$$

Now introduce Fourier transforms of $f^{(1)}$ and $E^{(1)}$:

$$f^{(1)}(x,u,t) = \frac{1}{2\pi}\int dk e^{-ikx} f^{(1)}(k,u,t) \tag{28}$$

$$E^{(1)}(x,t) = \frac{1}{2\pi}\int dk e^{-ikx} E^{(1)}(k,t). \tag{29}$$

Thus:

$$\begin{aligned}\int dk e^{-ikx} f^{(1)}(k,u,t) = \int dk e^{-ik(x-ut)} f^{(1)}(k,u,0) - \\ \frac{q}{m}\frac{\partial f^{(0)}}{\partial u}\int_0^t d\tau \int dk e^{ik(x+u(\tau-t)} E^{(1)}(k,\tau)\end{aligned} \tag{30}$$

Hence:

$$f^{(1)}(k,u,t) = e^{-ikut}f^{(1)}(k,u,0) - \frac{q}{m}\frac{\partial f^{(0)}}{\partial u}\int_0^t d\tau e^{iku(\tau-t)}E^{(1)}(k,\tau) \tag{31}$$

The Fourier transformed Poisson equation yields:

$$\begin{aligned}\frac{\partial E^{(1)}(x,t)}{\partial x} = ikE^{(1)}(k,t) = \omega_p^2 \int du \frac{\partial f^{(0)}}{\partial u}\int_0^\tau d\tau E^{(1)}(k,\tau)e^{iku(\tau-t)} - \\ 4\pi q n_0 \int due^{ikut}f^{(1)}(k,u,0)\end{aligned} \tag{32}$$

where $\omega_p^2 = \frac{4\pi n_0 q^2}{m}$. We can then define the following operators:

$$K(k,t-\tau) = \frac{\omega_p^2}{ik}\int due^{-iku(t-\tau)}\frac{\partial f^{(0)}}{\partial u} \tag{33}$$

$$S(k,t) = \frac{-4\pi q n_0}{ik}\int due^{-ikut}f^{(1)}(k,u,0) \tag{34}$$

such that

$$S(k,t) = E^{(1)}(k,t) + \int_0^t d\tau K(k,t-\tau)E^{(1)}(k,t) \tag{35}$$

This relation is readily solved with a Laplace transform:

$$E^{(1)}(k,p) = \int_0^\infty dte^{-pt}E^{(1)}(k,t) \tag{36}$$

$$E^{(1)}(k,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dpe^{pt}E^{(1)}(k,p) \tag{37}$$

where σ is defined such that there are no singularities in the integrand for Re $p > \sigma$. Using convolution theorem, we can extrapolate the following:

$$K(k,p)E^{(1)}(k,p) = \int_0^\infty dte^{-pt}[\int_0^\tau d\tau K(k,t-\tau)E^{(1)}(k,t)] \tag{38}$$

$$E^{(1)}(k,p)[1+K(k,p)] = S(k,p) \tag{39}$$

$$E^{(1)}(k,p) = \frac{S(k,p)}{D(k,p)} \tag{40}$$

where the dispersion function $D(k,p)$ is defined as:

$$D(k,p) = 1 + \frac{\omega_p^2}{ik}\int du \frac{\partial f^{(0)}}{\partial u}\frac{1}{p+iku} \tag{41}$$

$$E^{(1)}(x,t) = \frac{1}{2\pi}\int dke^{ikx}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{dp}{2\pi i}e^{pt}\frac{S(k,p)}{D(k,p)} \tag{42}$$

$$f^{(1)}(k,u,t) = e^{-ikut}f^{(1)}(k,u,0) - \frac{q}{m}\frac{\partial f^{(0)}}{\partial u}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{dp}{2\pi i}\frac{e^{pt}}{p+iku}E^{(1)}(k,p) \quad (43)$$

$$S(k,p) = \frac{-4\pi q n_0}{ik}\int du\frac{f^{(1)}(k,u,0)}{p+iku} \quad (44)$$

We proceed by dividing through by ik so as to force the denominator of the integrand into the form $u-z$:

$$D(k,p) = 1 + \frac{\omega_p^2}{k^2}\int du\frac{f'^{(0)}(u)}{u-\frac{ip}{k}} \quad (45)$$

$$S(k,p) = \frac{-4\pi q n_0}{k^2}\int du\frac{f^{(1)}(k,u,0)}{u-\frac{ip}{k}} \quad (46)$$

In order to return to Cartesian coordinates, we must perform an inverse Laplace transform, which requires separating the real and imaginary parts from $D(k,p)$ and $S(k,p)$:

$$I(z) = \int_{-\infty}^{\infty} du\frac{g(u)}{u-z},\ \text{Im }(z) > 0 \text{ where } z = \frac{ip}{k} \quad (47)$$

If we assume that g(u) is an entire function of u, we must redefine the function $I(z)$ in order for it to be analytically continuous into the lower half of the z plane.

$$\tilde{I}(z) = \begin{cases} \int_{-\infty}^{\infty} du\frac{g(u)}{u-z}, & \text{Im }(z) > 0 \\ \int_{-\infty}^{\infty} du\frac{g(u)}{u-z} + 2\pi i g(z), & \text{Im }(z) < 0 \end{cases} \quad (48)$$

Since $\tilde{I}(z)$ is analytically continuous across the entire z-plane, we can use contour integration to evaluate the integral. It is important to note that the contour always passes under the pole where $u = z$

$$D(k,p) = 1 + \frac{\omega_p^2}{k^2}\int_C du\frac{f'^{(0)}(u)}{u-\frac{ip}{k}} \quad (49)$$

$$S(k,p) = \frac{-4\pi q n_0}{k^2}\int_C du\frac{f^{(1)}(k,u,0)}{u-\frac{ip}{k}} \quad (50)$$

The above mentioned pole is located at the zeros of the dispersion relation. These singularities represent the collective response of a plasma to a perturbation and occur at:

$$D(k,p_i) = 0,\ i = 1,2,...,N \quad (51)$$

In order to determine the rate of decay at the above mentioned singularities, we must define $p = -i\omega + \gamma$, $z = \frac{\omega}{k} + \frac{i\gamma}{k}$ and $z_0 = \frac{\omega}{k}$. These re-defined variables are used to Taylor expand $D(k,p)$ about $z = z_0$. The first order real part of the expansion is:

$$k^2 = \omega_p^2\ P\int_{-\infty}^{\infty} du\frac{f'^{(0)}(u)}{u-\frac{\omega}{k}} \quad (52)$$

where P is Cauchy's principle value. If we take $\frac{\omega}{k} >> v_{th}$, the dispersion relation can be expanded to:

$$P\int_{-\infty}^{\infty} du\frac{f'^{(0)}(u)}{u-\frac{\omega}{k}} = -\frac{k}{\omega}[\int_{-\infty}^{\infty} f'^{(0)}(u)(1-\frac{ku}{\omega})^{-1}] = -\frac{k}{\omega}\sum_{n=0}^{\infty}(\frac{k}{\omega})^n\int_{-\infty}^{\infty} du\cdot f'^{(0)}(u)\cdot u^n \tag{53}$$

If we define the thermal velocity of the electrons in the plasma as $v_{th}^2 \equiv \int_{-\infty}^{\infty} du\cdot u^2 f^{(0)}(u)$ we can determine the frequencies ω_k of the electrostatic waves that correlate with the zeros of the dispersion relation.

$$k^2 = \omega_p^2[\frac{k^2}{\omega^2}\int_{-\infty}^{\infty} du\cdot f^{(0)}(u) + 3\frac{k^4}{\omega^4}\int_{-\infty}^{\infty} du\cdot f^{(0)}(u)\cdot u^2 + ...] \tag{54}$$

$$\omega_k^2 \simeq \omega_p^2 + 3k^2 v_{th}^2 + ..., \text{ where } \omega_p^2 = \frac{4\pi n_0 q^2}{m_e} \tag{55}$$

If we expand Eq. (49) to $O(\gamma)$ we obtain the damping constant γ_k that correlates with the zeros of the dispersion relation.

$$\gamma_k = \frac{\pi\omega_k^3}{2k^2}f'^{(0)}(\frac{\omega_k}{k}) \tag{56}$$

As can be seen, the slope of the particle distribution function determines whether or not the wave is damped or unstable. In order to relate this back to the damping of the electric field, we find that the perturbed electric field decays with a rate γ_k:

$$E(k,t) \simeq e^{\gamma_k t} \tag{57}$$