Multi-Field Inflation with a Curved Field-Space Metric

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by

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Abstract

We studied the nature of the curvature of a field-space given a perturbation of the field-space metric in an inflation scenario with a Dante's Inferno potential. We found the scalar curvature of the field-space to be negative with respect to the Gaussian curvature. Finally we recommend that future work be done on the nature of cosmological observables given the curvature in the field-space.

Chapter 1

Introduction

1.1 Why Inflation

Cosmological inflation was developed in order solve some problems in traditional Big Bang cosmology, in which cosmological observations brought into question how a traditional Big Bang universe could have produced some of the characteristics observed. The two main problems inflation solves are the Horizon problem and the Flatness problem.

The horizon problem arises from the fact that, overall, the Cosmic Microwave Background (CMB), looks very uniform [1][1], as seen in Figure 1.1. This uniformity is problematic under Big Bang cosmology. In that model, at each moment while the universe is expanding, regions that were not causally connected come into contact, which would produce a non-homogeneous CMB [1]. This connection is caused by the domination of the latest time component in the equation for the particle horizon [1]. Beyond this, even causally disconnected regions are homogeneous [1].

The flatness problem describes the fact that the curvature of the universe currently has an Ω_k value of less than 10^{-3}, which is very small, suggesting that the universe is very flat [1]. The magnitude of the curvature also has to increase over time, meaning that in the very early stages of the universe, it was even flatter than it is now. Why

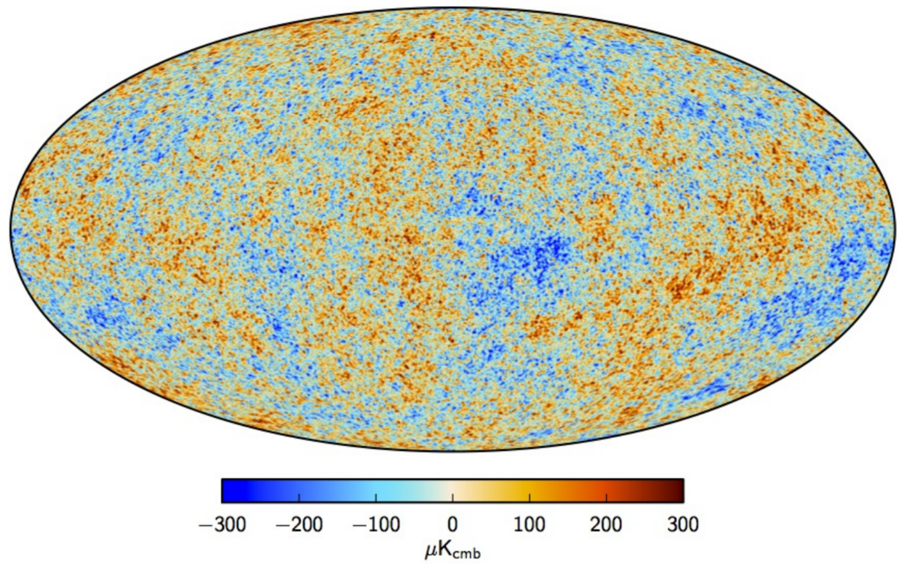

Figure 1.1: This plot shows the temperature fluctuations in the Cosmic Microwave Background as measured in 2015 by the Planck experiment [4].

could this be [1]? One option is that the universe simply chose a value of near zero for curvature, however, this is unsatisfactory and seems unlikely [1].

Inflation provides relatively straightforward solutions to both of these problems. For the Horizon problem, inflation allows for what seem to be causally disconnected regions across the sky to have been causally connected prior to inflation [1]. Then, since during inflation the universe expanded by such a large factor in such a short amount of time, the seemingly causally connected areas ended up extremely far apart [1]. Inflation solves the flatness problem as if the universe did expand by such a large magnitude in such a short time, that today the universe appears flat from our observations, just as the surface of the earth appears flat at the scales humans are used to, even if it is known that on a grander scale there is curvature.

Chapter 2

Inflationary Theory

2.1 Inflaton

Inflation theory is centered on the idea of the inflaton, some field which exists throughout the universe which drove inflation. The inflaton has some potential, and different potentials produce different models of inflation with different mechanics. The inflaton is defined by its lagrangian, which in this case is the following [2]:

$$\mathcal{L} = \sum_{i=1,j=1}^{N} G_{ij}(\phi)\partial_\mu\phi_i\partial_\nu\phi_j g^{\mu\nu} - V(\phi)$$

2.2 Slow-Roll Inflation

In slow-roll inflation the inflaton has a nearly flat potential that drops off at the end of inflation. The end of inflation is defined to be when the slow-roll parameter ϵ reaches 1 [1]. During the inflationary period, the universe expanded by roughly 60 e-folds (factors of e) [1]. The potential of the inflaton informs the values of the three slow-roll parameters, which, in turn, are used to calculate cosmological observables. The slow roll parameters are a function of the potential of the inflaton (which is found as a function of I, the value of the inflaton) and its derivatives [2].

$$\epsilon = \frac{M_{PL}^2}{2}(\frac{V'(I)}{V})^2$$

$$\eta = M_{PL}^2 \frac{V''(I)}{V}$$

$$\gamma = M_{PL}^4 \frac{V'(I)V'''(I)}{V^2}$$

2.3 Multi-Field Inflation

Multi-field inflation is motivated by a constraint known as the Lyth Bound. The Lyth Bound states that, if inflation is driven by a single field, then that field would have to take values at the Planck scale or higher in order to produce primordial gravitational waves with a high power [3]. To get around this limit, one can define the inflaton as being the combination of two or more fields [3].

2.4 Observables

There are several cosmological observables one can calculate using the slow-roll parameters and the potential. First is the ratio of amplitudes in tensor modes to those in scalar modes in the primordial density fluctuations, which describes the power in primordial gravitational waves (I_i is the value of the inflaton at the start of inflation, 50-60 e-folds before the end of inflation) [2]:

$$\tilde{r} = [16\epsilon]_{I=I_i}$$

Next is the scalar tilt, which describes the scale invariance of scalar modes in the primordial density perturbations [2]:

$$n_s = [1 + 2\eta - 6\epsilon]_{I=I_i}$$

The running of the scalar tilt, which describes how the scalar tilt change across scales [2]:

$$n_r = [16\epsilon\eta - 24\epsilon^2 - 2\gamma]_{I=I_i}$$

Following that is the scalar amplitude, which is the amplitude in the power spectrum of scalar modes in the primordial density perturbations [2]:

$$\Delta_R^2 = [\frac{V}{24\pi^2\epsilon}]_{I=I_i}$$

The number of e-folds during inflation, which is the number of factors of e the universe expanded by during inflation [2]:

$$N_e = \int_{I_i}^{I_f} \frac{V}{V'(I)} dI$$

There are known values for some of these quantities that are used to check the accuracy of potential inflationary theories. The number of e-folds of inflation is usually set somewhere between 50 and 60, the scalar amplitude is set to be $\Delta_R^2 = 2.2 * 10^{-9}$ Mpc$^-$1, the scalar tilt is set to be $n_s = 0.96$ (the latter two both come from the Planck experiment), and $\epsilon = 1$ at the end of inflation [4].

Chapter 3

Curvature

3.1 The Field-Space Metric

The field-space metric describes the space in which the inflaton field (or fields) exist. Normally the field-space is flat, however the field space used in this project introduces a curvature. The metric being used in this project is the following:

$$g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + ar^2 \end{bmatrix}$$

3.2 Riemann Curvature Tensor

Everything one can know about the curvature of a given space can be found using the Riemann curvature tensor [5]:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^m_{jl}\Gamma^i_{mk}$$

The Γ^i_{kl} terms are Christoffel symbols of the second kind and the m index is summed over all indices (in this case, r and θ) for each term [5][6]:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} (\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m})$$

Given the field-space metric being used in this model, the necessary Christoffel symbols come out to the following:

$$\Gamma^r_{rr} = 0$$

$$\Gamma^r_{r\theta} = 0$$

$$\Gamma^r_{\theta r} = 0$$

$$\Gamma^\theta_{rr} = 0$$

$$\Gamma^r_{\theta\theta} = -ar$$

$$\Gamma^\theta_{r\theta} = \frac{ar}{(1 + ar^2)}$$

$$\Gamma^\theta_{\theta r} = \frac{ar}{(1 + ar^2)}$$

$$\Gamma^\theta_{\theta\theta} = 0$$

3.3 Ricci Tensor

From the Riemann curvature tensor R^i_{jkl}, one can contract the i and k indices in order to get the Ricci tensor R_{jl} [7]. Using the symmetry relations and our knowledge from the given field-space metric, we found the following Ricci tensors:

$$R_{rr} = \frac{a}{(1 + ar^2)^2}$$

$$R_{r\theta} = 0$$

$$R_{\theta r} = 0$$

$$R_{\theta\theta} = \frac{a}{(1 + ar^2)}$$

3.4 Curvature Scalar

Finally, one can calculate the curvature scalar, R, of a given space, where [8]:

$$R_{ij} = g_{ij} R$$

Therefore:

$$R = g^{ij} R_{ij}$$

Which, using the given field space metric is:

$$R = g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta}$$

$$R = \frac{a}{(1+ar^2)^2} + \frac{1}{(1+ar^2)} \frac{a}{(1+ar^2)}$$

$$R = \frac{2a}{(1+ar^2)^2}$$

This result can then by checked using the following equation:

$$R_{ab} = \frac{1}{2} g_{ab} R$$

This check was performed and was successful as is shown:

$$R_{rr} = \frac{1}{2} g_{rr} R \ = \frac{1}{2}(1) \frac{2a}{(1+ar^2)^2} = \frac{a}{(1+ar^2)^2}$$

$$R_{\theta\theta} = \frac{1}{2} g_{\theta\theta} R \ = \frac{1}{2}(1+ar^2) \frac{2a}{(1+ar^2)^2} = \frac{a}{(1+ar^2)^2}$$

This confirms the value found for the scalar curvature R. Given the form used for the Riemann tensor, the scalar curvature has a negative relation to the Gaussian curvature K. The Gaussian curvature is defined as positive for a sphere given the conventions we used. The relation of the scalar curvature to the Gaussian is the following. One can see how R varies with r and a within the range that ar^2 is small and acts as a perturbation in figure 1.

$$R = -2K$$

Since the scalar curvature of the field space is positive for all positive a, the overall curvature of the field space is negative, or hyperbolic.

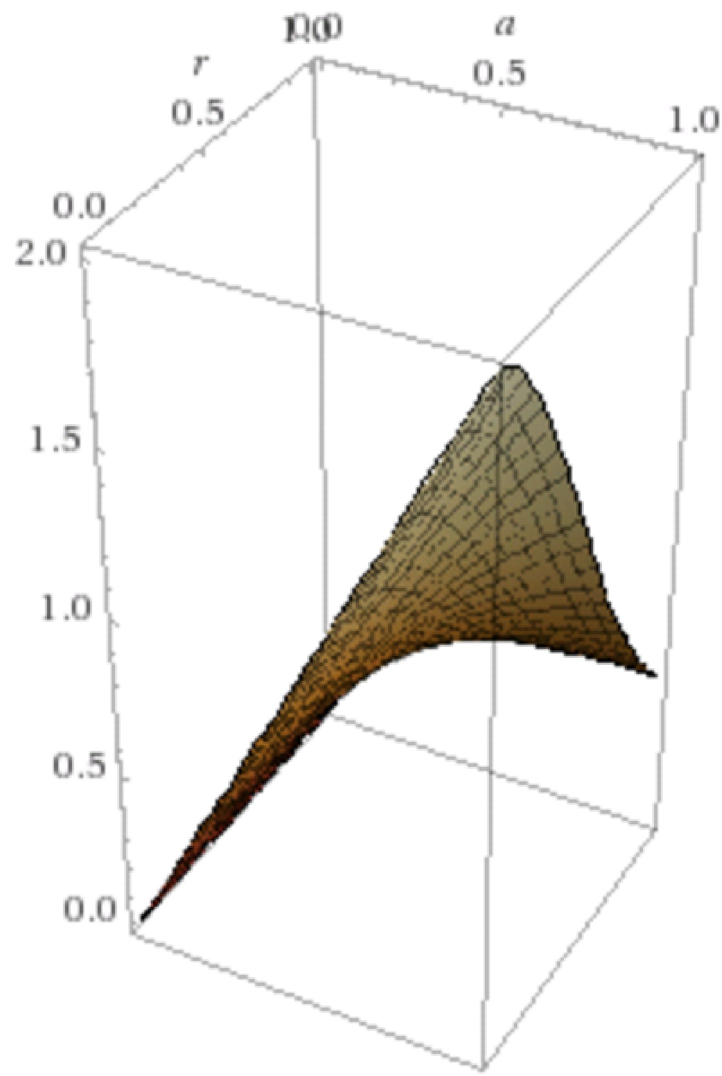

Figure 3.1: Scalar curvature is plotted along the vertical axis, while r and a are plotted on the two horizontal axes.

Chapter 4

Inflationary Calculations

4.1 The Potential

In this project the Dante's Inferno potential was chosen to be the potential for the inflaton. Dante's Inferno is a two-dimensional potential with r and θ components [2].

$$V(r,\theta) = W(r) + \Lambda^4[1 - cos(\frac{r}{f_r} - \frac{\theta}{f_\theta})]$$

The potential has a trench that goes down in a spiral, down which the inflaton rolls until the end of inflation. In order to perform the calculations necessary to describing the cosmological observables given this potential, we needed to describe the potential as a single-field approximation. This single field we call I. Therefore, we needed to find both r and θ as functions of I. We know that along the trench that $\frac{\delta V}{\delta r} \approx 0$. From this, we know that $r \approx \theta \frac{f_r}{f_\theta}$ along the trench, this approximation can help us find V(I).

With these approximations, we then found an approximation of the Lagrangian along the trajectory of the trench.

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + \dot{\theta}^2(1 + ar^2)) - V(r(\theta(I)), \theta(I))$$

$$\mathcal{L} = \frac{1}{2}(\theta^2(\frac{f_r^2}{f_\theta^2} + 1) + \dot{\theta}^2(1 + a\theta^2 \frac{f_r^2}{f_\theta^2})) - V(r(\theta(I)), \theta(I))$$

Which we defined as the following:

$$\mathcal{L} = \frac{1}{2}\dot{I}^2 - V(r(\theta(I)), \theta(I))$$

From this we can then find $\frac{dI}{d\theta}$ which we can integrate in order to solve for $I(\theta)$ which can be inverted to find $\theta(I)$ in order to solve for V(I).

$$\frac{dI}{d\theta} = \sqrt{(\frac{f_r^2}{f_\theta^2} + 1 + a\theta^2 \frac{f_r^2}{f_\theta^2})}$$

$$I(\theta) = \int \sqrt{(\frac{f_r^2}{f_\theta^2} + 1 + a\theta^2 \frac{f_r^2}{f_\theta^2})} d\theta$$

Integrating analytically, we found $I(\theta)$ as the following:

$$I(\theta) = \frac{\left[\sqrt{\frac{af_r^2\theta^2 + f_r f_\theta + f_\theta^2}{f_\theta^2}}\left(\sqrt{a}f_r\theta\sqrt{af_r^2\theta^2 + f_r f_\theta + f_\theta^2} + f_\theta(f_r + f_\theta)log\left(\sqrt{a}\sqrt{af_r^2\theta^2 + f_r f_\theta + f_\theta^2} + af_r\theta\right)\right)\right]}{\left[2\sqrt{a}f_r\sqrt{af_r^2\theta^2 + f_r f_\theta + f_\theta^2}\right]}$$

Chapter 5

Conclusions and Future Work

We found that given this perturbation in a field space metric for a multi-field inflation model, the curvature in the metric will increase negatively as a increases positively. Thus, the field-space will have hyperbolic curvature if a is positive. For future work, we would recommend numerically integrating dI over $d\theta$ to attempt to get a clearer definition of $I(\theta)$. We would then recommend inverting $I(\theta)$ equation in order to obtain $\theta(I)$ such that one could then plug in $\theta(I)$ into $V(r(\theta(I)), \theta(I))$ in order to obtain V(I). Then, given V(I), we recommend calculating derivatives of V in order to obtain the slow-roll parameters, ϵ, η, and γ. With the slow-roll parameters, one could then calculate the various cosmological observables. A research goal for future projects could be to plot the various observables in relation to the curvature, a, in order to see how the curvature would affect the observables.

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