Nondiffracting Waves in 2D and 3D

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by

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Abstract

Nondiffracting waves are waves with intensity profiles that do not change as they propagate. This paper explores the diffraction behavior of truncated nondiffracting waves in both two and three dimensions. These truncated versions of nondiffracting waves can be shown to propagate farther than Gaussian waves of comparable central widths, suggesting that the truncated waves can also resist diffraction.

1 Introduction

Most waves will diffract, or spread out, as they propagate. This paper explores certain types of waves that do not diffract as they propagate. This means that the wavefront has the same pattern regardless of how far it has propagated. Some may find it surprising that there exists a nondiffracting wave other than the plane wave. In fact, such a wave can be "beam-like": its intensity is concentrated in a specific spot and decays to zero away from that spot. This paper explores nondiffracting waves in both three and two dimensions.

A true nondiffracting wave requires infinite space and energy, so these cannot exist in practice. This paper considers what happens if a finite portion of a nondiffracting wave is produced. Rayleigh-Sommerfeld diffraction theory can be used to calculate the propagation of these waves and observe their diffractive behavior.

Durnin [1] analyzed the behavior of such waves in three dimensions, but he did not consider two-dimensional waves in his paper. This paper describes how Rayleigh-Sommerfeld diffraction theory is used to explore the diffraction of the "truncated" nondiffracting waves that Durnin studied, along with the two-dimensional counterparts these waves.

2 Background

Most types of beams exhibit diffraction as they propagate. A beam-like wave has an intensity that is initially concentrated around the axis of propagation. So if Huygens' principle is applied, every point on such a wavefront will behave as the source of a spherical wave. The result of this is the spreading-out of the beam as it propagates forward, which is known as diffraction.

If a wave is nondiffracting, then it does not exhibit this spreading as it propagates. In other words, the intensity distribution in the transverse plane is the same no matter how far the wave has propagated [1]. A trivial example of a nondiffracting wave is the infinite plane wave. A plane wave will look the same regardless of how far it propagates. However, if the plane wave passes through a finite aperture, the resulting wave will begin to diffract according to Huygens' principle.

A 1987 paper written by J. Durnin [1] presents another type of beam that does not diffract. This wave has the interesting property that the intensity distribution in the transverse plane has a peak at the axis of propagation and decays to zero as distance from the axis of propagation increases. Therefore, this wave resembles a beam, yet it does not spread out as it propagates. Durnin found this type of beam by solving the wave equation for a scalar field E:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E(\boldsymbol{r}, t) = 0, \tag{1}$$

with a monochromatic time dependence and with the requirement that the z-dependence be $e^{i\beta z}$, where β is constant. The solution Durnin found has azimuthal symmetry:

$$E(\boldsymbol{r}, t) = e^{i(\beta z - \omega t)} J_0(\alpha\rho), \tag{2}$$

where $\alpha^2 + \beta^2 = \left(\frac{\omega}{c}\right)^2$, and $\rho^2 = x^2 + y^2$ if this wave is traveling in the z-direction. So the intensity pattern of this beam follows a Bessel function of the first kind, which has a peak in the center, and rings surrounding this peak that decrease in intensity as distance from the central peak increases. The time average of the intensity of this beam satisfies the following property:

$$I(\rho, z \geq 0) = I(\rho, z = 0). \tag{3}$$

In other words, the intensity profile of this wave at any point in the direction of propagation is exactly the same as the initial intensity profile [1]. Therefore, this wave is nondiffracting.

However, this Bessel beam is truly nondiffracting only when the beam has been produced over an infinite plane, which would require infinite space and energy. A finite aperture can be used to produce a truncated Bessel beam, which is not be perfectly nondiffractive, but maintains its intensity profile better than other types of beams, such as Gaussian beams.

Durnin's paper explores only this three-dimensional solution, but a two-dimensional analogue can be found by solving the wave equation in 2D for a monochromatic, nondiffracting wave. Let z be the direction of propagation, and let x be the transverse direction. Since the wave is monochromatic, the wave equation becomes the Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right) \psi(x)\, e^{i(k_z z - \omega t)} = 0, \tag{4}$$

where $\alpha^2 = k^2 - k_z^2$, and $k = \frac{\omega}{c}$. The solution for $\psi(x)$ is $\psi(x) = \cos(\alpha x + \phi)$, where ϕ is a phase. So the nondiffracting solution in two dimensions is:

$$\Psi(x, z, t) = e^{i(k_z z - \omega t)} \cos(\alpha x + \phi). \tag{5}$$

The next section will introduce Rayleigh-Sommerfeld diffraction theory, a mathematically explicit way of implementing Huygens' principle for propagation of a wave that passes through an aperture in a plane (or a line, in the two-dimensional case).

3 Rayleigh-Sommerfeld diffraction theory

An equation is needed that describes the wave function at an arbitrary observation point r' given the source wave function at a point r in the aperture.

3.1 3D case

Let z be the direction of propagation, and let the aperture be in the $z = 0$ plane. Consider a function $\psi(r)$ that satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\,\psi(r) = 0 \tag{6}$$

and an appropriate Green's function that satisfies

$$(\nabla^2 + k^2)\,G(r - r') = \delta^3(r - r') \tag{7}$$

in the region $V = \{(x, y, z) \mid z > 0\}$. The function $G(r - r') = -\frac{1}{4\pi}\frac{e^{ik|r-r'|}}{|r-r'|}$ satisfies this condition. Since $\psi(r) = \int_V \psi(r')\,\delta^3(r - r')\,\mathrm{d}^3r$, it follows that

$$\begin{aligned}
\psi(r') &= \int_V \psi(r)\,(\nabla^2 + k^2)\,G(r - r')\,\mathrm{d}^3r && (8)\\
&= \int_V \psi(r)\,\nabla^2\,G(r - r')\,\mathrm{d}^3r + \int_V k^2\,\psi(r)\,G(r - r')\,\mathrm{d}^3r. && (9)
\end{aligned}$$

Using integration by parts on the first integral,

$$\begin{aligned}
\int_V \psi\,\nabla^2 G\,\mathrm{d}^3r &= \int_S \psi\,\nabla G\,\mathrm{d}\mathbf{S} - \int_V \nabla\psi\,\nabla G\,\mathrm{d}^3r && (10)\\
&= \int_S \psi\,\nabla G\,\mathrm{d}\mathbf{S} - \int_S (\nabla\psi)\,G\,\mathrm{d}\mathbf{S} + \int_V (\nabla^2\psi)\,G\,\mathrm{d}^3r, && (11)
\end{aligned}$$

where S is the boundary of V. The third integral in equation (11) combines with the second integral in equation (9) to form $\int_V G\,(\nabla^2 + k^2)\,\psi\,\mathrm{d}^3 r$, which is zero. Therefore,

$$\psi(r') = \int_S (\psi\,\nabla G - G\,\nabla\psi)\,\mathrm{d}\mathbf{S}. \tag{12}$$

This integrand is nonzero only at the aperture, so the integral becomes an integral over the area of the aperture at $z = 0$. Huygens' principle relies on a wave disturbance at $z = 0$ that becomes the source for the propagating wave. The derivative of the source wave function $\nabla\psi$ is not necessarily known, so a Green's function is needed that is zero in the area of the aperture, which would make $\nabla\psi$ irrelevant in this integral. If $r' = (x', y', z')$, then let $\tilde{r}' = (x', y', -z')$. Let $R = |r - r'|$ and $\tilde{R} = |r - \tilde{r}'|$. Then consider the Green's function

$$G(r - r') = -\frac{1}{4\pi}\frac{e^{ikR}}{R} + \frac{1}{4\pi}\frac{e^{ik\tilde{R}}}{\tilde{R}}, \tag{13}$$

which is related, but not identical, to the Green's function given after equation (7). This function satisfies the condition in equation (7) for the region V. This is because $(\nabla^2 + k^2)G$ gives the sum of two Dirac delta functions: one centered at r' in the region V, and one centered at $\tilde{r}'$ outside the region V. Since this is zero at every point other than r' and $\tilde{r}'$, the Dirac delta function outside the region V does not have an effect in the region V. Also, since $R = \tilde{R}$ at $z = 0$, it follows that $G(r - r') = 0$ at $z = 0$. Therefore, the second term in the integrand of equation (12) is zero in the region being integrated over. If $\mathrm{d}\mathbf{S} = \hat{n}\,\mathrm{d}S$, where $\hat{n}$ is an inward-pointing normal, then equation (12) becomes

$$\psi(r') = \int_A \psi\,\frac{\partial G}{\partial z}\,\mathrm{d}A, \tag{14}$$

where A is the aperture, and

$$\frac{\partial G(r-r')}{\partial z} = -\frac{1}{2\pi}\left(ik - \frac{1}{R}\right)\left(\frac{z}{R}\right)\frac{e^{ikR}}{R}. \tag{15}$$

So the equation that can be used to describe the wave function at an observation point in V is

$$\psi(r') = -\frac{1}{2\pi}\int_A \psi(x,y,z=0)\left(ik - \frac{1}{R}\right)\left(\frac{z}{R}\right)\frac{e^{ikR}}{R}\,\mathrm{d}A. \tag{16}$$

If R is much larger than the size of the aperture, then equation (15) can be reduced to

$$\frac{\partial G}{\partial z} \approx -\frac{ik}{2\pi}\frac{z}{R}\frac{e^{ikR}}{R}. \tag{17}$$

Typically, $R \gg k$ for the purposes of this paper, so this assumption is valid for the calculations done in Section 4. So an approximation that can be used to describe the wave function at a point in V is

$$\psi(r') \approx -\frac{ik}{2\pi}\int_A \psi(x,y,z=0)\frac{e^{ikR}}{R}\frac{z}{R}\,\mathrm{d}A. \tag{18}$$

3.2 2D case

Let z and z' be the propagation-direction coordinates of the aperture point and the observation point, respectively. Let x and x' be the transverse-direction coordinates. The process for deriving the analogous equation in the two-dimensional case is similar to the three-dimensional case, and a similar result is obtained:

$$\psi(x',z') = \int_{-l}^{l}\left(\psi(x,z=0)\frac{\partial G}{\partial z} - G\frac{\partial \psi}{\partial z}\right)\mathrm{d}x, \tag{19}$$

where $2l$ is the length of the aperture.

A Green's function that satisfies

$$(\nabla^2 + k^2)\, G(r - r') = \delta^2(r - r') \tag{20}$$

is $G(r - r') = \frac{1}{4} N_0(kR)$, where $r = (x, z)$, $r' = (x', z')$, and $R = |r - r'|$. Since $(\nabla^2 + k^2) J_0(kR) = 0$, the function $G(r - r') = \frac{-i}{4}(J_0(kR) \pm i N_0(kR))$ also satisfies equation (20). Since only outgoing waves are desired, choose

$$G(r - r') = \frac{-i}{4}(J_0(kR) + i N_0(kR)) = \frac{-i}{4} H_0^{(1)}(kR). \tag{21}$$

As in the three-dimensional case, the derivative $\frac{\partial \psi}{\partial z}$ may not be known, so a Green's function is needed that makes the second term in the integral of equation (19) zero at the aperture. Therefore, consider the Green's function

$$G(r - r') = \frac{-i}{4} H_0^{(1)}(kR) + \frac{i}{4} H_0^{(1)}(k\tilde{R}), \tag{22}$$

where $\tilde{R} = |r - \tilde{r}'|$ and $\tilde{r}' = (x', -z')$. Since this is the sum of two delta functions, one in the region $\{z > 0\}$ and one in the region $\{z < 0\}$, it follows that the Green's function in equation (22) satisfies the condition in equation (20) for the region $\{z > 0\}$. Since $R = \tilde{R}$ when $z = 0$, the Green's function is zero at the aperture, so the second term in the integral in equation (19) is zero. Therefore, the integral becomes

$$\psi(x', z') = \int_{-l}^{l} \psi(x, z = 0)\, \frac{\partial G}{\partial z}\, \mathrm{d}x, \tag{23}$$

which gives the result

$$\psi(r') = \frac{k}{2i} \int_{-l}^{l} \psi(x, z = 0)\, H_1^{(1)}(kR)\, \frac{z'}{R}\, \mathrm{d}x. \tag{24}$$

Since kR is very large for the purposes of this paper, the asymptotic form of the Hankel function can be used, which gives the following approximation:

$$\psi(r') \approx \frac{k}{2i} \int_{-l}^{l} \psi(x, z = 0) \sqrt{\frac{2}{\pi kR}}\, e^{i\left(kR - \frac{3\pi}{4}\right)}\, \frac{z'}{R}\, \mathrm{d}x. \tag{25}$$

4 Long-distance propagation

4.1 3D waves

Numerical calculations can be used to determine how these "truncated" Bessel beams propagate. In the following image, Rayleigh-Sommerfeld diffraction theory was used to calculate the propagation of the Bessel beam. Figure 1 shows how the intensity of the central peak of the Bessel beam changes as it propagates in the z-direction. It propagates with good strength for about one meter with some oscillation in central intensity before decaying to zero.

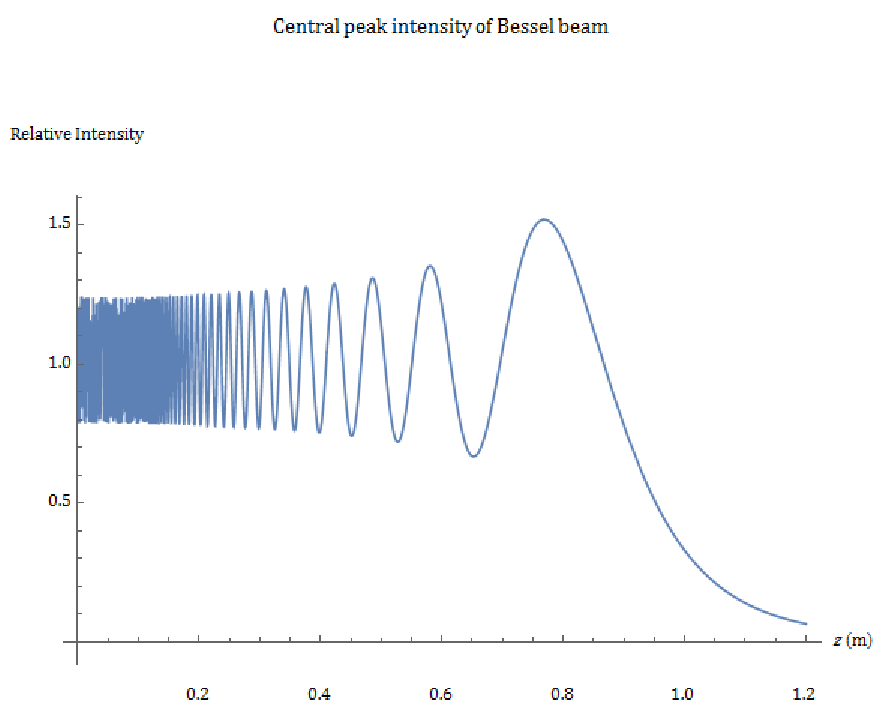

Figure 1: Intensity of the central peak ($\rho = 0$) as a function of the distance propagated z (in meters). This is for a Bessel beam with $\alpha = 240.5\,\text{cm}^{-1}$, a circular aperture of radius $2\,\text{mm}$, and wavelength $\lambda = 0.5\,\mu\text{m}$.

Figure 2 shows how the intensity profile, the transverse distribution of intensity, of the Bessel beam changes as it propagates. The beam maintains its shape well, even after traveling 75 cm, but with some decay at the outer part of the beam. However, after traveling 100 cm, the central peak has begun to decay significantly.

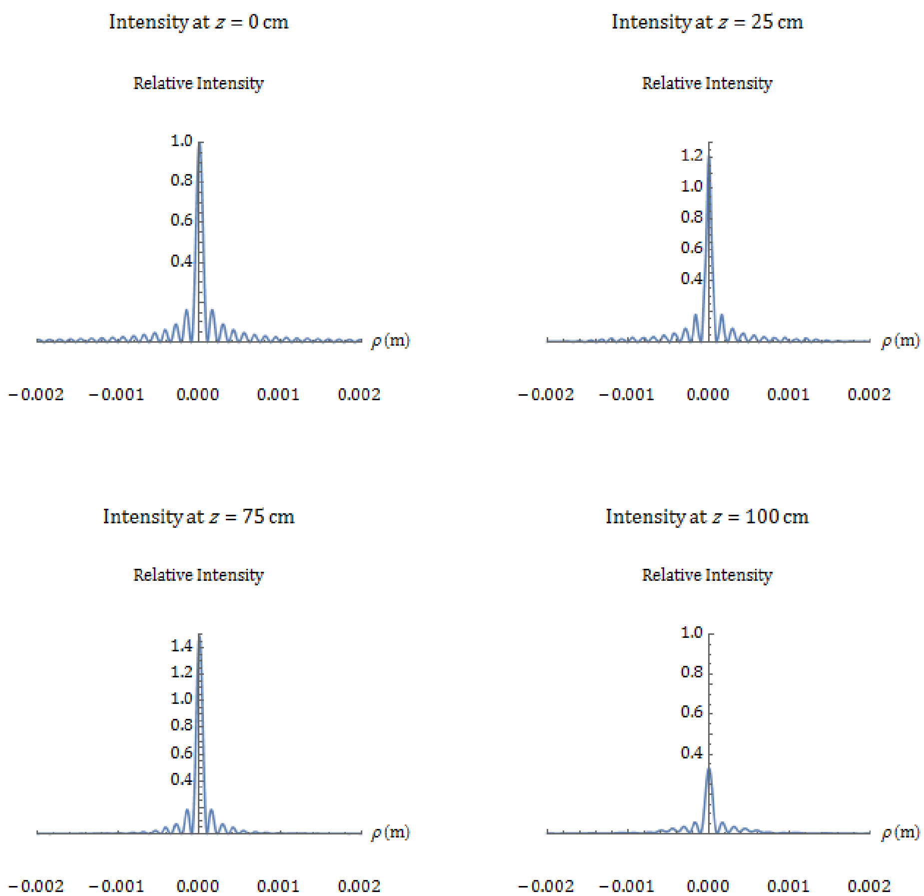

Figure 2: Transverse intensity profiles of the Bessel beam at $z = 0, 25, 75$, and 100 cm. This is for a Bessel beam with $\alpha = 240.5\,\text{cm}^{-1}$, a circular aperture of radius 2 mm, and wavelength $\lambda = 0.5\,\mu$m.

For comparison, figure 3 shows a plot of how a Gaussian beam width the same wavelength and a full with at half-maximum of $100\,\mu$m propagates. This width is chosen so that it matches the width of the central peak of the Bessel beam that was shown in figure 2.

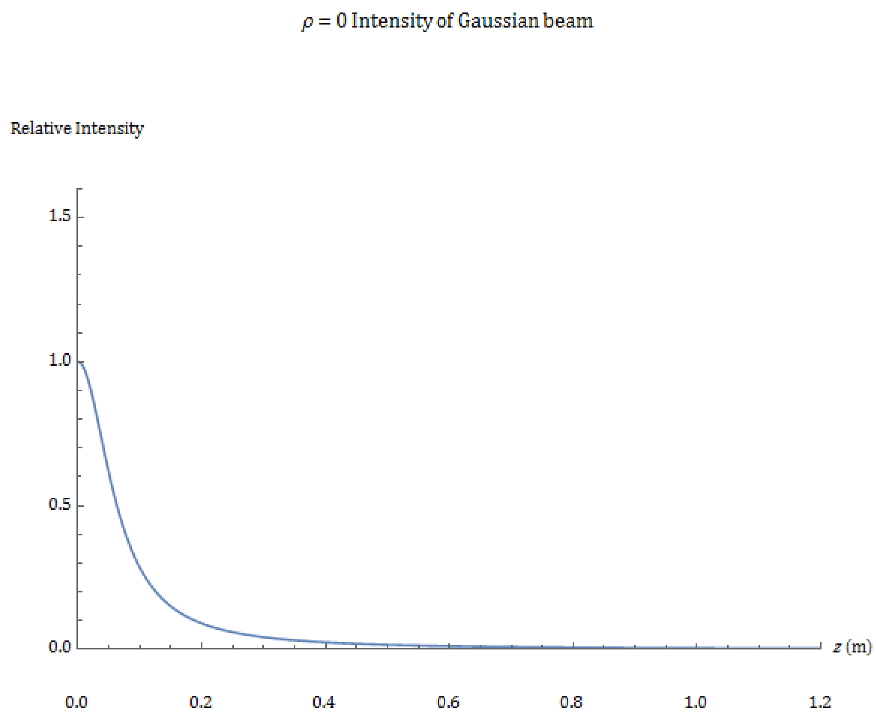

Figure 3: Intensity of the central peak ($\rho = 0$) as a function of the distance propagated z (in meters). This is for a Gaussian beam with FWHM $= 100\,\mu$m and wavelength $\lambda = 0.5\,\mu$m.

Figure 4 shows how the intensity profile of the Gaussian beam changes as it propagates. The beam starts to decay and spread out after a relatively short propagation distance.

As shown by these plots, the intensity at $\rho = 0$ of the Bessel beam oscillates around its starting value, but persists for almost a meter of propagation for the selected parameters, while the intensity of the Gaussian beam begins to decay very quickly. Therefore, even a truncated Bessel beam resists diffraction for almost one meter, much better than a Gaussian beam. Experimental work done by Durnin, Miceli, and Eberly [2] has confirmed that a Bessel beam does propagate in a manner

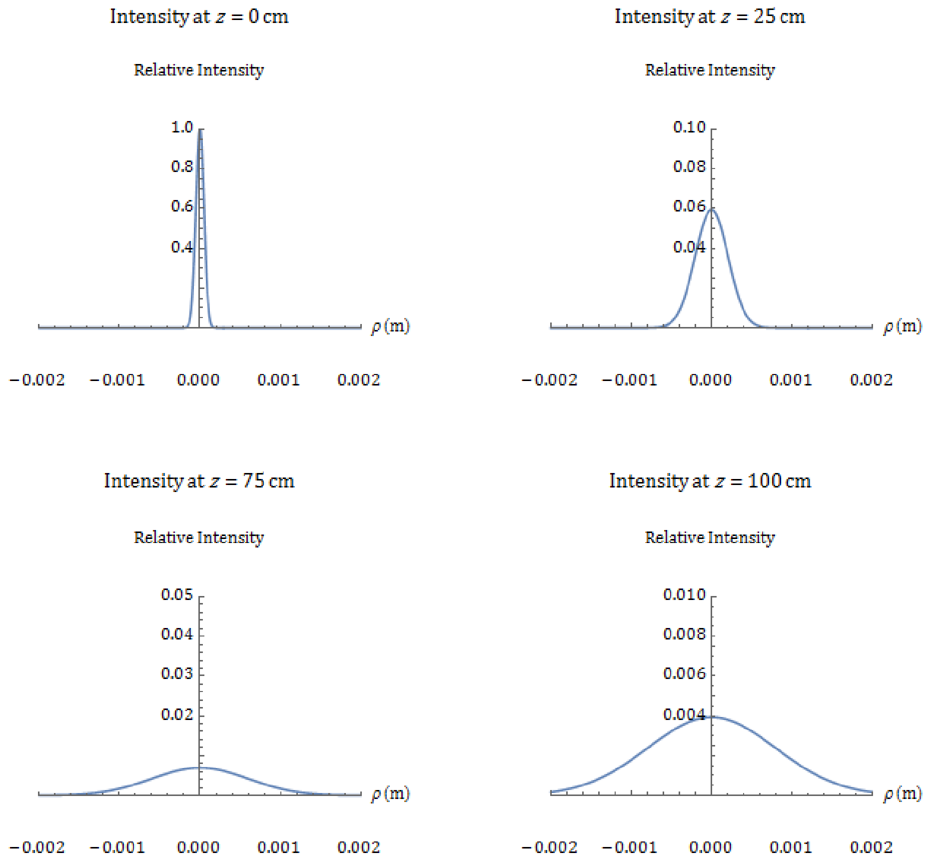

Figure 4: Transverse intensity profiles of the Gaussian beam at $z = 0, 25, 75,$ and $100\,\text{cm}$. This is for a Gaussian beam with FWHM $= 100\,\mu\text{m}$ and wavelength $\lambda = 0.5\,\mu\text{m}$.

predicted by these numerical calculations.

4.2 2D waves

Numerical calculations were also used to test the propagation of a truncated version of the two-dimensional nondiffracting wave, the cosine wave. Figure 5 shows how the intensity of the $x = 0$ peak of a cosine wave propagates. It behaves in a manner very similar to the truncated Bessel beam in three dimensions, with its intensity oscillating near its starting value for nearly one meter before quickly decaying to zero. This suggests that a truncated cosine wave resists diffraction in the same way that the previously discussed Bessel beam resists diffraction.

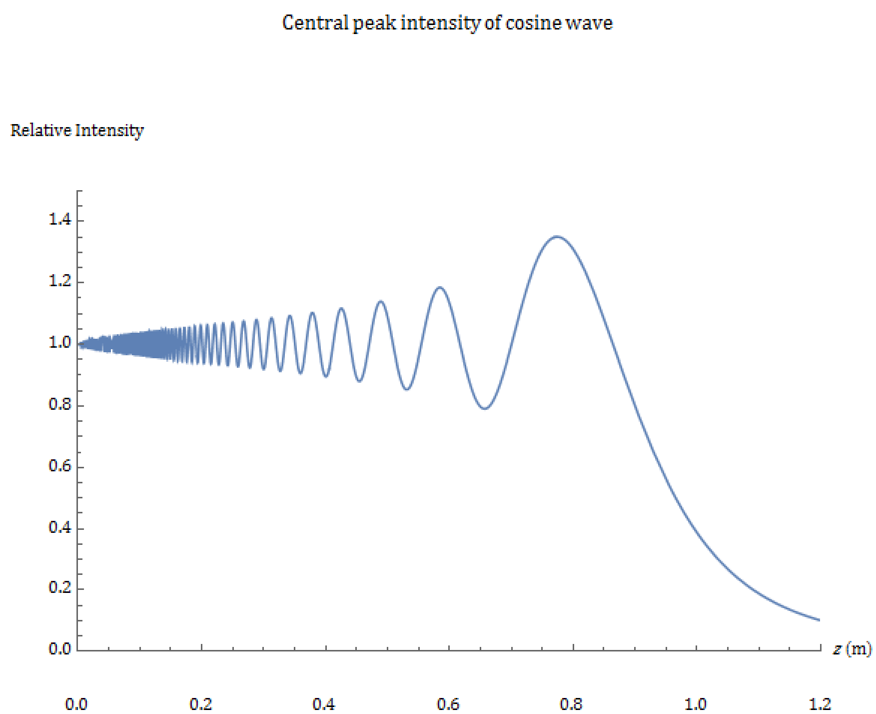

Figure 5: Intensity of the central peak ($x = 0$) as a function of the distance propagated z (in meters). This is for a 2D cosine wave described by $u(x, 0) = \cos(\alpha x)$, where $\alpha = 240.5\,\mathrm{cm}^{-1}$, passing through an aperture with length $4\,\mathrm{mm}$. This wave has wavelength $\lambda = 0.5\,\mu\mathrm{m}$.

Figure 6 shows how the intensity profile of the cosine wave changes as it propagates. As in the three-dimensional case, the shape of the center of the cosine wave is well maintained for a certain propagation distance, while the outermost parts of the wave decay and noticeably spread out as the wave propagates.

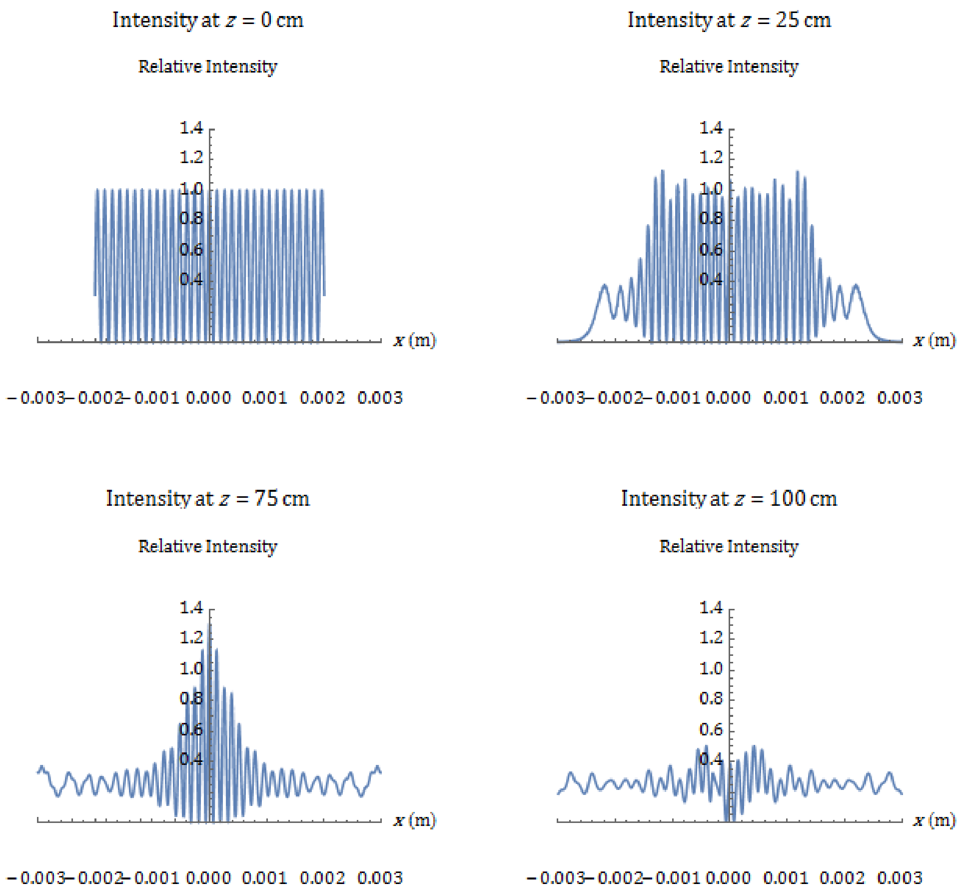

Figure 6: Transverse intensity profiles of the two-dimensional cosine wave at $z = 0, 25, 75,$ and $100\,\text{cm}$. This is for a 2D cosine wave described by $u(x, 0) = \cos(\alpha x)$, where $\alpha = 240.5\,\text{cm}^{-1}$, passing through an aperture with length $4\,\text{mm}$. This wave has wavelength $\lambda = 0.5\,\mu\text{m}$.

For comparison, consider a two-dimensional Gaussian beam that has a full width at half maximum that matches the central peak of the cosine wave. The propagation of this beam is shown in figure 7. As in the three-dimensional example, this beam decays to zero very quickly, much faster than the cosine wave. One may say this is because, unlike in the cosine wave, the Gaussian beam only has one peak. Therefore, there are no other parts of the beam that can "protect" the central peak from diffraction by feeding energy into the center from the wings of the beam.

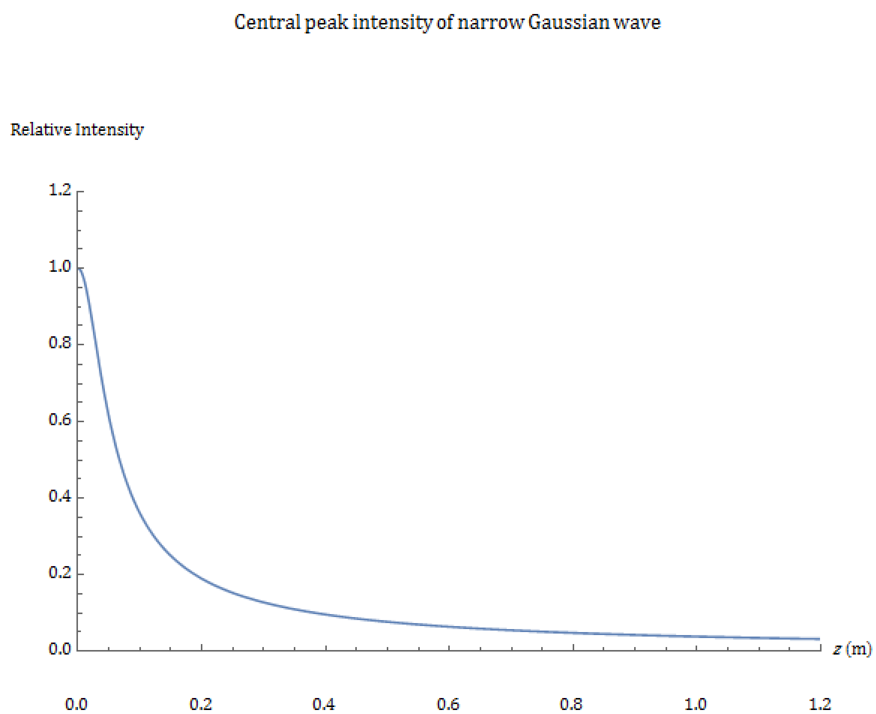

Figure 7: Intensity of the central peak ($\rho = 0$) as a function of the distance propagated z (in meters). This is for a Gaussian beam with FWHM $= \frac{\pi}{240.5}$ cm and wavelength $\lambda = 0.5\,\mu$m. The width for this Gaussian was chosen to match the width of a single oscillation of the cosine wave.

In an attempt to find a Gaussian wave that propagates farther, consider a Gaussian wave that has a full width at half maximum of approximately one-half the size of the aperture, as shown in figure 8.

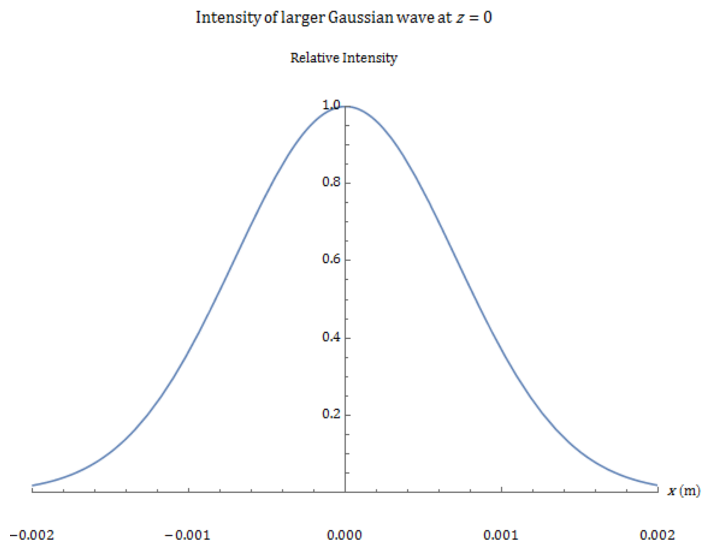

Figure 8: Intensity at $z = 0$ of a Gaussian wave described by $u(x, 0) = e^{\frac{-x^2}{2c^2}}$, where $c = 1\,\text{mm}$.

This larger Gaussian wave propagates much farther than either the smaller Gaussian or the cosine wave, as shown in figure 9. However, the shape of this Gaussian begins to change rapidly as a result of diffraction. This can be seen in figure 10, which shows the intensity profile beginning to fracture after the wave has traveled 75 cm.

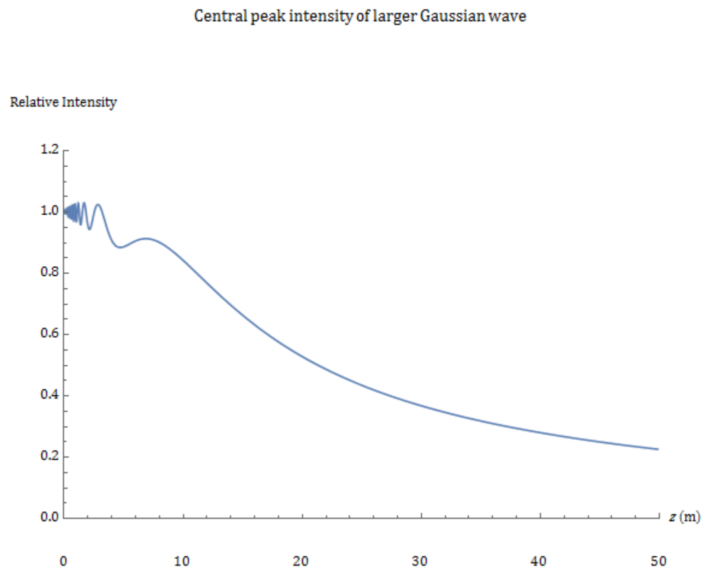

Figure 9: Intensity of the central peak ($x = 0$) as a function of the distance propagated z (in meters). This is for the Gaussian wave described by $u(x, 0) = e^{\frac{-x^2}{2c^2}}$, where $c = 1\,\text{mm}$, passing through an aperture with length $4\,\text{mm}$. This wave has wavelength $\lambda = 0.5\,\mu\text{m}$.

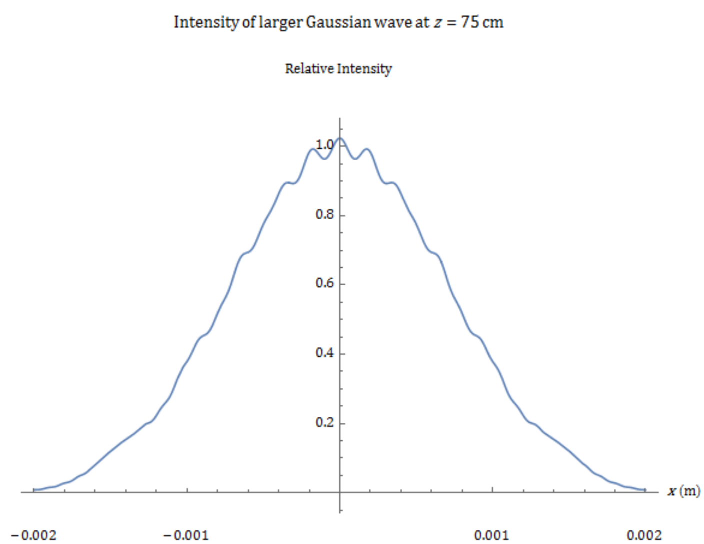

Figure 10: Intensity at $z = 75\,\text{cm}$ of the Gaussian wave described by $u(x, 0) = e^{\frac{-x^2}{2c^2}}$, where $c = 1\,\text{mm}$.

5 Conclusion

Numerical simulations of the Bessel beams discovered by Durnin show that they can propagate for long distances without diffracting, even if their transverse profiles are cut off by passing through a finite opening. Experiments have confirmed this [2]. A three-dimensional Bessel beam was shown to propagate much farther and with less diffraction than a Gaussian beam of comparable central width, even with only a small portion of the beam passing through a finite aperture. For a two-dimensional nondiffracting wave, a cosine wave, it was shown that a truncated version can propagate relatively far while resisting diffraction. The behavior of this wave was quite similar to the behavior of the Bessel beam in three dimensions. So, as in the three-dimensional case, a truncated, two-dimensional, nondiffracting wave retains some of the nondiffractive properties of the full, non-truncated wave.

The Gaussian waves used for comparison were constructed so that the full width at half maximum matched the central peak of the Bessel beam in three dimensions or the cosine wave in two dimensions. However, if the width of the Gaussian is significantly larger, say close to the size of the aperture, then the Gaussian can propagate much farther than the nondiffracting wave. This larger Gaussian does experience distortions as a result of diffraction, which changes the shape of the wave, even at the center of the peak. The truncated nondiffracting waves, on the other hand, are better able to resist changing shape at the central peak, so even though a large Gaussian can propagate farther, the Bessel beam and cosine wave have the advantage of maintaining the shape of the center of the wave.

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