Sums of Convergent Series

I. Introduction

Often the functions used in the sciences are defined as infinite series. Determining the convergence or divergence of a series becomes important and it is helpful if the sum of a convergent series can be determined. Most of the tests taught in Chapter 11 can determine convergence but do not allow us to find the sum of the series. One exception is a convergent geometric series which has the sum,

\[ s = \frac{a}{1 - r} \]  

(1)

where \( a \) is the first term of the series and \( r \) is the common ratio.

For any series, if there is a formula for the \( n^{th} \) partial sum, \( s_n \), the exact sum is

\[ s = \lim_{n \to \infty} s_n \]  

(2)

if the limit exists. If a series is a telescoping series, we can find a formula for \( s_n \) and use (2) to determine its convergence and sum.

Series that converge may or may not have a known sum, however. For many series, we can only estimate the sum.

II. Estimating Sums

Given a convergent series, any \( n^{th} \) partial sum may be used to approximate the sum of the series. Some helpful calculator steps are at the end of this lab.

Example 1: We know the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. How would you support this conclusion?
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots, \text{ is convergent and one possible approximation of the sum is } s_4 = 1.4236 \bar{1}. \text{ Of course, a better approximation is } s_6 = 1.4913\bar{8} \text{ and } s_{100} = 1.6349839\ldots, \text{ is better still. In fact, we know the } n^{th} \text{ partial sum becomes a better approximation as } n \text{ increases. Why?}
\]

III. Error

Every approximation is just that... an approximation rather than the exact value. So every approximation has an associated error. In the previous example, the first four terms of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) are used to calculate \( s_4 \), this approximation has an error equal to \( a_5 + a_6 + a_7 + \ldots \).

Generally, the error in using \( s_n \) to approximate the sum of a convergent series is equal to the sum of all the remaining terms. That is,

\[
\text{Error} = \text{actual sum} - \text{approximate sum}
\]

or

\[
\text{Error} = s - s_n
\]

This error is called the remainder which makes sense as it is made up of the remaining terms not used to calculate the approximation, \( s_n \). The notation \( R_n \) denotes the error in using \( s_n \) to approximate \( s \). So,

\[
R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots \quad (3)
\]

If we cannot find the exact sum of a series, then we certainly cannot find the exact remainder associated with an approximation. However, if the remainder can be bound, we can improve the approximation of the sum. If the Integral Test can be used to determine convergence of a series, it can also be used to find bounds for \( R_n \). The sketches below illustrate how \( R_n \) is bound between two area calculations.
So the error is bound as follows:

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$  \hspace{1cm} (4)$$

The exact sum is somewhere between the approximation plus the smallest error and the approximation plus the largest error. That is,

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx$$  \hspace{1cm} (5)$$

If an Alternating Series is convergent, the error in approximating the sum with $s_n$ can also be bound. The error is no greater than the absolute value of the first “neglected term” in $s_n$.

Example 2: The alternating harmonic series shown below is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$$

If $s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$ is used to approximate the sum of the series, we know the remainder is equal to the sum of all the remaining terms. Once again, we cannot find the exact remainder but we *can* find an upper bound for the remainder. Notice what happens if we approximate the remainder, adding one more term in each approximation:

$R_5 \approx -\frac{1}{6}$

$R_5 \approx -\frac{1}{6} + \frac{1}{7} = -\frac{1}{42}$

$R_5 \approx -\frac{1}{6} + \frac{1}{7} - \frac{1}{8} = -\frac{25}{168}$

We see $-\frac{1}{6} > -\frac{1}{42} > -\frac{25}{168}$. The error continues to get smaller and smaller, as more terms are added. Thus, the error (in absolute value) will be no greater than $b_6 = \frac{1}{6}$.

Generally, for series $\sum_{n=1}^{\infty} (-1)^{n-1}b_n$ that satisfy the conditions of the Alternating Series Test, the size of the error is

$$|R_n| = |s - s_n| \leq b_{n+1}$$  \hspace{1cm} (6)$$
IV. Helpful Calculator Stuff

We can list values of the sequence of terms and the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$.

To clear y= and lists: 2nd, MEM, 4, enter

Bring up table as shown: STAT, ENTER

Arrow over and/or up to highlight $L_1$: 2nd, LIST, OPS, seq(x,x,1,100,1) ENTER

This displays the $n$-values from 1 to 100 in $L_1$.

Arrow right and up to highlight $L_2$. Enter the $a_n$ formula, replacing $n$ with $L_1$, then ENTER.

This displays the sequence of terms in the series.

Arrow right and up to highlight $L_3$. Enter: 2nd, LIST, OPS, 6, ENTER, 2nd, $L_2$, ENTER

This displays the sequence of partial sums of the series.

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>terms</td>
<td>partial sums</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>
Math 112  S20 Lab 8 Exercises       Name: ___________________________Section: ____Score: ____

You may use your textbook, lab and notes. Students may work cooperatively but must submit his/her own set of Lab Exercises. No calculators unless otherwise noted.

1. Consider the series \( \sum_{n=1}^{\infty} \frac{1}{n(n + 3)} \).

(a) Use partial fraction decomposition to express \( a_n \) as a combination of two fractional expressions.

(b) State what type of series this is.

(c) Expand the series to find a formula for \( s_n \).

(d) Find the sum of the series, if it exists. (Hint: this involves finding a limit.)
2. Consider the series, $\sum_{k=1}^{\infty} \frac{3^{k-1}}{4^{k+1}}$. (a) Expand the series to 5 terms.

(b) State what type of series this is.

(c) Determine if the series converges or diverges. Your work should include steps to justify your conclusion.

(d) If the series converges, find the exact sum if this is possible.

3. (a) Show the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges by using the Alternating Series Test.

(b) Approximate the sum of the series using $s_6$. (calculator)

(c) Find the largest error expected in using this approximation for the sum. (fraction or decimal)
4. (a) Show the series $\sum_{n=1}^{\infty} ne^{-2n^2}$ satisfies the required conditions for the Integral Test.

(b) Use the Integral Test to show the series converges.

(c) Find the largest error expected in using the 4th partial sum to approximate the sum of this series. (Final answer should be an exact value, no calculators.)