1. Find each limit using limit laws and analytic methods (not tables of values or graphs). Identify indeterminate form(s) present and the use of l'Hospital's Rule where applicable. If the limit does not exist, justify this and state DNE.

(6) (a) $\lim_{x\to\infty} x\cos\left(\frac{1}{x}\right) = \infty \cdot 1 = \infty$ DNE

$\infty \cdot 1$

$\frac{1}{x} \to 0$ as $x \to \infty$

$\therefore \cos\left(\frac{1}{x}\right) \to 1$

(6) (b) $\lim_{x\to 0^-}\left[10^{\ln(x+1)} + e^{\frac{1}{x}}\right] = 1 + 0 = 1$

$x+1 \to 1$ as $x \to 0^-$

$\ln(x+1) \to 0$ as $x \to 0^-$

$10^{\ln(x+1)} \to 1$ as $x \to 0^-$

$1/x \to -\infty$ as $x \to 0^-$

$e^{1/x} \to 0$ as $x \to 0^-$

(6) (c) $\lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1} = \frac{1}{e}$

$1-x \to 0^-$ $\therefore \frac{1}{1-x} \to -\infty$

form: $1^{+\infty}$

let $y = x^{\frac{1}{1-x}}$

$\ln y = \ln x^{\frac{1}{1-x}}$

$\ln y = \frac{1}{1-x} \cdot \ln x$

$\ln y = \frac{\ln x}{1-x}$

$\lim_{x\to 1^+} \frac{\ln x}{1-x} \overset{L}{=} \lim_{x\to 1^+} \frac{1/x}{-1}$

$\frac{0}{0}$

$= \lim_{x\to 1^+} -\frac{1}{x} = -1$

(6) (d) $\lim_{x\to 0^+}\left(\frac{\sin 3x}{x}\right)^{100} = \left[\lim_{x\to 0^+} \frac{\sin 3x}{x}\right]^{100} \overset{L}{=} \left[\lim_{x\to 0^+} \frac{3\cos 3x}{1}\right]^{100} = \left[3\cos(0)\right]^{100}$

$\frac{0}{0}$

$= 3^{100}$

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(6) 2. *Show the use of the definition of continuity* and find the real numbers r and s such that the function

$$f(x)=\begin{cases} x+r, & x\le 0 \\ \dfrac{x^2-9}{x-3}, & 0<x<3 \\ s, & x\ge 3 \end{cases} \quad \text{is continuous on } (-\infty,\infty).$$

$$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$$

$$\lim_{x\to 0^-} x+r = r = \lim_{x\to 0^+} \frac{x^2-9}{x-3} \qquad \lim_{x\to 3^-} \frac{x^2-9}{x-3} = s = \lim_{x\to 3^+} s$$

$$r = r = \frac{-9}{-3} = 3 \qquad \lim_{x\to 3^-} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = s$$

$$r = 3 \qquad 6 = s$$

(6) 3. *Use the definition of the derivative* to find $f'(x)$ if $f(x)=\sqrt{x+1}$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \frac{(\sqrt{x+h+1}+\sqrt{x+1})}{(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+\cancel{h}\cancel{+1}-\cancel{x}\cancel{-1}}{\cancel{h}(\sqrt{x+\underset{0}{h}+1}+\sqrt{x+1})} = \frac{1}{2\sqrt{x+1}}$$

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4. Use the rules and techniques of differentiation to find the following derivatives. Some functions require logarithmic or implicit differentiation. Do not use the limit definition to find any of the derivatives. Simplify your final answer.

(6) (a) Find y' if $y = \dfrac{2x+5}{\sqrt{3-x}}$.

$$y' = \frac{\sqrt{3-x}\,(2) - (2x+5)\cdot\frac{1}{2}(3-x)^{-1/2}(-1)}{3-x} = \frac{\left[2\sqrt{3-x} + \frac{2x+5}{2\sqrt{3-x}}\right]2\sqrt{3-x}}{(3-x)\,2\sqrt{3-x}}$$

$$= \frac{4(3-x) + 2x+5}{2(3-x)^{3/2}} = \frac{17-2x}{2(3-x)^{3/2}}$$

(6) (b) Find $\dfrac{dy}{dx}$ if $\sin(x)\cos(y)+y^2=1$.

$$\sin(x)(-\sin(y))\cdot y' + \cos(y)\cos(x) + 2yy' = 0$$

$$y'(2y - \sin(x)\sin(y)) = -\cos(x)\cos(y)$$

$$y' = \frac{-\cos(x)\cos(y)}{2y - \sin(x)\sin(y)}$$

-OR-

$$y' = \frac{\cos(x)\cos(y)}{\sin(x)\sin(y) - 2y}$$

(6) (c) Find $f'(x)$ if $f(x) = x^3 \sin^2(5x)$.

$$f'(x) = x^3 \cdot 2\sin(5x)\cos(5x)\cdot 5 + \sin^2(5x)\cdot 3x^2$$

$$f'(x) = 10x^3\sin(5x)\cos(5x) + 3x^2\sin^2(5x)$$

-OR-

$$x^2\sin(5x)\left[10x\cos(5x) + 3\sin(5x)\right]$$

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(d) Find the derivative of $y = x^{x-\arctan x}$.

(6)

$$\ln y = \ln x^{x-\tan^{-1}x}$$

$$\ln y = (x-\tan^{-1}x)\cdot \ln x$$

$$\frac{1}{y}\cdot y' = (x-\tan^{-1}x)\cdot\frac{1}{x} + \ln x\left(1-\frac{1}{1+x^2}\right)$$

$$y' = y\left[\frac{x-\tan^{-1}x}{x} + \ln x - \frac{\ln x}{1+x^2}\right]$$

$$y' = x^{(x-\tan^{-1}x)}\left[1-\frac{\tan^{-1}x}{x} + \ln x - \frac{\ln x}{1+x^2}\right]$$

(6) (e) Find $\frac{dy}{dx}$ if $y = \sin^{-1}(e^{-3x})$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^{-3x})^2}}\cdot e^{-3x}(-3)$$

$$\frac{dy}{dx} = \frac{-3e^{-3x}}{\sqrt{1-e^{-6x}}}$$

(6) 5. Use the Intermediate Value Theorem and Mean Value Theorem to show the equation $x^3+2x+1=0$ has *exactly* one real root.

① $f(x)$ is a polynomial $\therefore$ continuous on $\mathbb{R}$.

Note: $f(1) = 1+2+1 = 4 > 0$

$f(-1) = -1-2+1 = -2 < 0$

and $-2 < 0 < 4$ $\therefore$ $\exists$ some $c \in (-1,1)$ s.t. $f(c) = 0$ $\therefore$ a root exists in $(-1,1)$

② Suppose two roots, a and b with $a<b$ (in $(-1,1)$)

$f(x)$ is differentiable on $\mathbb{R}$. By MVT $\exists$ some $x=c \in [a,b]$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c) = 0$ ($f(a)=f(b)=0$)

But $f'(c) = 3c^2+2 \neq 0$ for any $x=c$ $\therefore$ only one root exists. □

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(6) 6. Find the equation of the line that is tangent to the curve $y = \sqrt[3]{x^2}$ at $x = 8$. $(\sqrt[3]{8})^2 = 2^2 = 4$

$y = x^{2/3}$ @ $(8, 4)$

$m: y' = 2/3\, x^{-1/3}$

$m = 2/3 \frac{1}{\sqrt[3]{8}} = (2/3)(\frac{1}{2}) = \frac{1}{3}$

$y - 4 = 1/3 (x - 8)$

– OR –

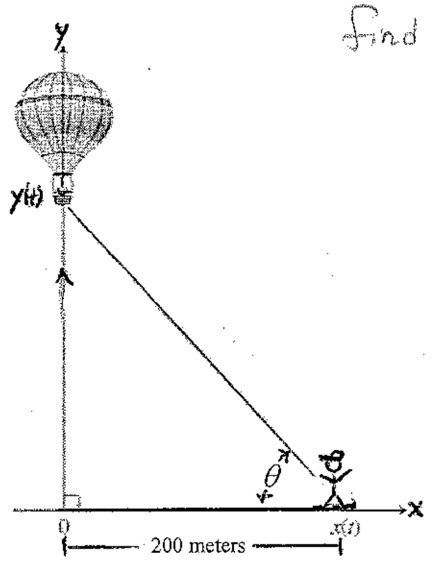
$y = 1/3 x - 8/3 + 12/3$

$y = 1/3 x + 4/3$

(10) 7. A balloon is rising vertically above a level, straight road. An observer is located 200 meters down the road from the point directly below the balloon as seen in the figure. He determines the angle of elevation, θ, between himself and the balloon is increasing at a constant rate of $\frac{\pi}{200} \frac{rad}{s}$ when the angle of elevation is $\frac{\pi}{4} rad$. Find how fast the balloon is rising at this time. Remember to include units in your final answer.

$\frac{d\theta}{dt} = \pi/200 \ \frac{rad}{s}$

find $\frac{dy}{dt}$ when $\theta = \pi/4$



$\tan\theta = \frac{opp}{adj}$

$\tan\theta = \frac{y}{200}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = 1/200 \frac{dy}{dt}$

$\frac{1}{\cos^2(\pi/4)} \cdot \frac{\pi}{200} \cdot 200 = \frac{dy}{dt}$

$\frac{1}{(\sqrt{2}/2)^2} \cdot \pi = 2\pi \ m/s = \frac{dy}{dt}$

– OR –

$\tan^{-1} \frac{y}{200} = \theta$

$\frac{1}{1 + \frac{y^2}{(200)^2}} \cdot \frac{1}{200} \frac{dy}{dt} = \frac{d\theta}{dt}$

when $\theta = \pi/4$
$y = 200$

$\frac{dy}{dt} = \pi/200 \cdot 200 \left(1 + \frac{y^2}{(200)^2}\right)$

$\frac{dy}{dt} = \pi \cdot \left(1 + \frac{200^2}{200^2}\right) = 2\pi \ m/s$

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8. Consider the following function and its derivatives: $f(x)=\dfrac{x^2}{x-1}$, $f'(x)=\dfrac{x^2-2x}{(x-1)^2}$, $f''(x)=\dfrac{2}{(x-1)^3}$

(1) (a) State the domain of $f(x)$. $D: \{x \mid x \neq 1\}$

(4) (b) *Evaluate the proper limit(s) to find any vertical asymptote(s)* and the behavior of the graph of $f(x)$ at those asymptote(s).

$\lim_{x\to 1^-} \dfrac{x^2}{x-1} = -\infty$ (1/2)

n: $x^2 \to 1$ as $x \to 1^-$

d: $x-1 \to 0^-$ as $x \to 1^-$

$\lim_{x\to 1^+} \dfrac{x^2}{x-1} = +\infty$ (1/2)

n: $x^2 \to 1$ as $x \to 1^+$

d: $x-1 \to 0^+$ as $x \to 1^+$

VA @ $x=1$ (1)

(3) (c) *Evaluate the proper limit(s) to find any horizontal asymptote(s)* and the end behavior of the graph of $f(x)$.

$\lim_{x\to\infty} \dfrac{x^2/x}{x/x - 1/x} = \lim_{x\to\infty} x = \infty$ (no HA) (1 1/2)

$\lim_{x\to -\infty} \dfrac{x^2/x}{x/x - 1/x} = \lim_{x\to -\infty} x = -\infty$ no HA (1 1/2)

(5) (d) Find the interval(s) of increase and the intervals of decrease for $f(x)$ and the coordinate pair(s) of any local maximum(s) and any local minimum(s). (1) local max @ $(0,0)$

(1) local min @ $(2,4)$

critical #'s

$\dfrac{x(x-2)}{(x-1)^2} = 0$ when $x=0$, $x=2$

dne when $x=1$ (1/2)

sign of $f'(x)$: $(-)(-)$ + | $(+)(-)$ − | $(+)(-)$ − | $(+)(+)$ +

$+ + + +$ | 0 $- - -$ | 1 $- - -$ | 2 $+ + + +$

$x=-1$, $x=1/2$, $x=3/2$, $x=3$

(1/2)

$f(x)$ is increasing on $(-\infty,0)\cup(2,\infty)$ (1/2) $f(x)$ is decreasing on $(0,1)\cup(1,2)$

(3) (e) Find the intervals of concavity and the coordinate pair(s) of any inflection points.

$\dfrac{2}{(x-1)^3}$ dne @ $x=1$

$(+)/(-)$ $- - - -$ | 1 | $+ + + +$ $(+)/(+)$

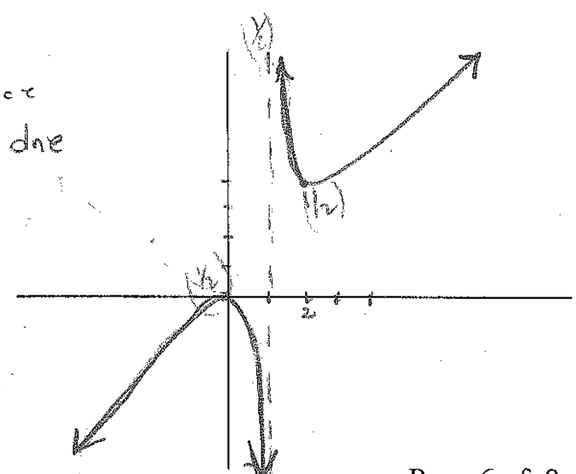
CD 0, CU 2

CD on $(-\infty,1)$ (1)

CU on $(1,\infty)$ (1)

no IP (1) since $f(1)$ dne

(2) (f) *Use the information from parts (a-e)* to sketch the graph of $f(x)$.



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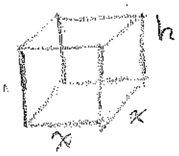
(10) 9. A closed rectangular container with a square base is to have a volume of $2000 in^3$. It costs \$2.00 per square inch for the top and bottom and \$1.00 per square inch for the sides. Find the dimensions of the cheapest container possible.

$V = x^2 h = 2000 \longrightarrow h = \frac{2000}{x^2}$

$\text{Cost} = 2 \cdot 2x^2 + 1 \cdot 4xh$



$C = 4x^2 + 4xh$

$C = 4x^2 + 4x \cdot \frac{2000}{x^2} \Rightarrow C = 4x^2 + \frac{8000}{x}$

x does not have a closed interval

$C'(x) = 8x - 8000x^{-2} = \frac{8x^3 - 8000}{x^2} = 0$ when $8(x^3 - 1000) = 0$

$x^3 = 1000$

$x = 10$

sign of C' $\quad \frac{(-)}{(+)} \quad \frac{(+)}{(+)}$

C' dne when $x = 0$

dec $\quad$ inc

$0 \quad x = 2 \quad 10 \quad x = 20 \quad x$

$\therefore$ $x = 10$ in is a minimum

$h = \frac{2000}{10^2} = \frac{2000}{100} = 20$ in

10 in $\times$ 10 in $\times$ 20 in

(6) 10. At time $t = 0$, a vehicle traveling with velocity $48\frac{ft}{s}$ begins to slow down with constant deceleration $a = -12\frac{ft}{s^2}$.
(a) Find the velocity of the vehicle, $v(t)$ at time t. (b) Find the distance traveled before the vehicle comes to a halt.

$a(t) = -12$

$v(t) = -12t + C \quad (v(0) = 48 \therefore C = 48)$

(a) $v(t) = -12t + 48$

(b) $s(t) = -6t^2 + 48t + K \quad (s(0) = 0 \therefore K = 0)$

$s(t) = -6t^2 + 48t$ $\quad$ vehicle stops when $v(t) = 0$

$-12t + 48 = 0$

$t = 4$ sec

$3 \cdot 16$

$s(4) = -6(4 \cdot 4) + 48(4)$

$= -6(16) + (3 \cdot 4 \cdot 16)$

$= 16[-6 + 12] = 16(6) = 96$ ft

(16)

11. Evaluate each integral.

(6) (a) $\int_1^9 \sqrt{\frac{3}{x}}\,dx$ $= \sqrt{3}\int_1^9 x^{-1/2}\,dx = \sqrt{3}\left[2x^{1/2}\right]_1^9 = 2\sqrt{3}\left[\sqrt{9}-\sqrt{1}\right]$
$= 2\sqrt{3}(2)$
$= 4\sqrt{3}$

(6) (b) $\int \frac{5}{1+x^2}\,dx$ $= 5\int \frac{1}{1+x^2}\,dx = 5\tan^{-1}x + C$

(6) (c) $\int_0^{3\pi/2} |\cos x|\,dx$ $= \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{3\pi/2}$

$$|\cos x| = \begin{cases} \cos x, & 0 \le x \le \pi/2 \\ -\cos x, & \pi/2 < x \le 3\pi/2 \end{cases}$$

$= \sin\pi/2 - \sin \cancel{0}^{\,0} - \sin 3\pi/2 + \sin \pi/2$
$1 - (-1) + 1 = 3$

(6) 12. Let $F(x)=\int_0^x \sqrt{3t^2+1}\,dt$. Find (a) $F'(2)$ and (b) $F''(2)$

$F'(x) = \sqrt{3x^2+1}$ ∴ $F'(2) = \sqrt{3(4)+1} = \sqrt{13}$

$F''(x) = \frac{1}{2}(3x^2+1)^{-1/2}(6x) = \frac{3x}{\sqrt{3x^2+1}}$

$F''(2) = \frac{3(2)}{\sqrt{3(4)+1}} = \frac{6}{\sqrt{13}}$

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(6) 13. *Use the limit definition of the definite integral*, $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, to evaluate $\int_0^2 (2x+1)\,dx$.

Note: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = a + i\Delta x = 0 + i\cdot\frac{2}{n}$

$x_i = \frac{2i}{n}$

$$\int_0^2 (2x+1)\,dx = \lim_{n\to\infty} \Delta x \cdot \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \frac{2}{n}\cdot\sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right)+1\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\cdot\frac{4}{n}\cdot\sum_{i=1}^{n} i + \frac{2}{n}\cdot\sum_{i=1}^{n} 1\right] = \lim_{n\to\infty}\left[\frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot n\right]$$

$$= \lim_{n\to\infty} 4\cdot\frac{n^2+n}{n^2} + 2 = 4 + 2 = 6$$

check $\int_0^2 2x+1\,dx = \frac{2}{2}x^2 + x\Big|_0^2 = 2^2 + 2 - 0$

$= 6$ ✓

(6) 14. Suppose that we estimate $\int_1^4 x^2\,dx$ by R_{1000}. That is, we use 1000 rectangles, of uniform width, with heights found at the right-endpoints. Then the base of the third rectangle (counting from the left) would extend from

$x_2 =$ 1.006 to $x_3 =$ 1.009, and the height of the third rectangle would be 1.018081

$\Delta x = \frac{4-1}{1000} = \frac{3}{1000} = .003$

$x_0 = 1$, $x_1 = 1 + .003 = 1.003$, $x_2 = 1 + 2(.003) = 1.006$

$f(x_3) = f(1.009)$

$= (1.009)^2 =$

(12) 15. Write TRUE if the statement is always true; otherwise write FALSE. Do not explain your answer choice.

TRUE (a) If $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$

FALSE (b) The function $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$.

FALSE (c) $\int_{-2}^{1} \frac{1}{x^2}\,dx = -\left(1+\frac{1}{2}\right) = -\frac{3}{2}$ $\frac{1}{x^2}$ is not continuous @ $x=0 \in [-2,1]$

FALSE (d) If $f''(a) = 0$, then f has an inflection point at $x = a$. Ex. $f(x) = x^4$, $f'(x) = 4x^3$

$f''(x) = 12x^2 = 0$ @ $x = 0$

but $x = 0$ is not IP

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