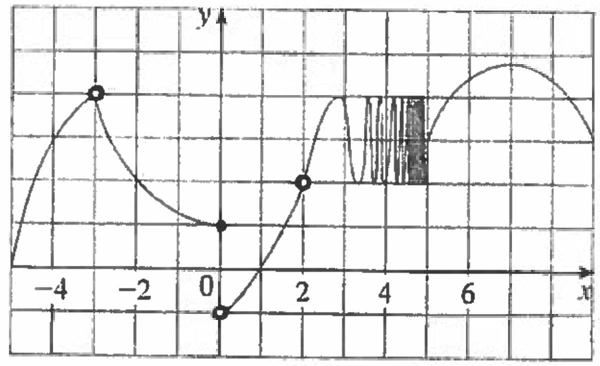
1. For the function h shown, state the value of each quantity. If it does not exist, state DNE and explain why.



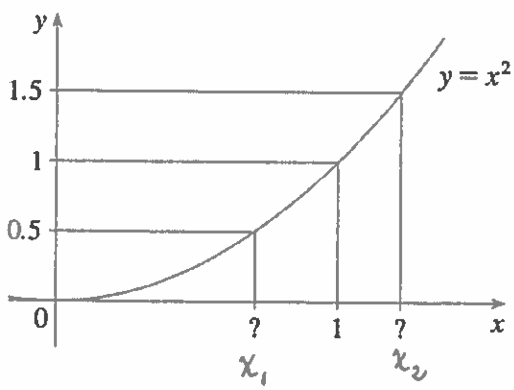
(a) $\lim_{x \to -3} h(x) =$ 4

(b) $h(-3) =$ DNE (no value here)

(c) $\lim_{x \to 0} h(x) =$ DNE since $\lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$

(d) $\lim_{x \to 5^-} h(x) =$ DNE since values (graph) oscillates

2. Below is the graph of $f(x) = x^2$ with $\lim_{x \to 1} f(x) = 1$ clearly identified. Find the largest number δ such that if $|x - 1| < \delta$ then $|x^2 - 1| < \frac{1}{2}$.



we need smaller of $|x_1 - 1|$ and $|x_2 - 1|$

for x_1: $\frac{1}{2} = x_1^2$

$x_1 = \sqrt{\frac{1}{2}}$ so $\left|\sqrt{\frac{1}{2}} - 1\right| < \delta$?

$\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}} = \frac{1-\sqrt{2}}{\sqrt{2}}$

for x_2: $\frac{3}{2} = x_2^2$

$x_2 = \sqrt{\frac{3}{2}}$ so $\left|\sqrt{\frac{3}{2}} - 1\right| < \delta$?

$\frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$

$\delta = \sqrt{\frac{3}{2}} - 1$

3. *Show the definition of continuity* in finding the constants a and b so that $f(x)$ will be continuous for all x.

$$f(x) = \begin{cases} x^2 + bx + 2, & x < 5 \\ 8, & x = 5 \\ ax + 3, & x > 5 \end{cases}$$

$f(x)$ is continuous for $x < 5$ and $x > 5$ since function is quadratic and linear, respectively.

For continuity at $x = 5$, we need

$$\lim_{x \to 5^-} x^2 + bx + 2 = f(5) = \lim_{x \to 5^+} ax + 3$$

$27 + 5b = 8$ and $5a + 3 = 8$

$b = -\frac{19}{5}$ $\qquad$ $a = 1$

4. (a – d) Find each limit, if it exists. If the limit does not exist, state DNE and explain why. If you use L'Hospital's rule, indicate this as well as the indeterminate form present. Justify any limit of $\pm\infty$.

(a) $\lim_{x\to 4^-} \frac{x^2+3x}{x^2-x-12}$ $= \lim_{x\to 4^-} \frac{x\cancel{(x+3)}}{\cancel{(x+3)}(x-4)} = -\infty$ (DNE)

since, $n: x \to 4$

$d: x-4 \to 0^-$

(b) $\lim_{x\to -1} \frac{\sqrt{4-x^2}-\sqrt{3}}{x+1}$ $\overset{L}{=} \lim_{x\to -1} \frac{1}{2}(4-x^2)^{-1/2} \cdot 2x = \lim_{x\to -1} \frac{-x}{\sqrt{4-x^2}} = \frac{1}{\sqrt{3}}$

$\frac{0}{0}$

– OR –

$= \lim_{x\to -1} \frac{(\sqrt{4-x^2}-\sqrt{3})(\sqrt{4-x^2}+\sqrt{3})}{(x+1)(\sqrt{4-x^2}+\sqrt{3})} = \lim_{x\to -1} \frac{4-x^2-3}{(x+1)(\sqrt{4-x^2}+\sqrt{3})}$, $(1-x)\cancel{(1+x)}$ $= \lim_{x\to -1} \frac{1-(-1)}{2\sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

(c) $\lim_{x\to -\infty} \frac{6x-3}{\sqrt{4x^2+3x}}$ $\frac{\cancel{x}}{-\sqrt{x^2}}$ $= \lim_{x\to -\infty} \frac{6-\cancel{\frac{3}{x}}^{0}}{-\sqrt{4+\cancel{\frac{3}{x}}^{0}}} = \frac{6}{-2} = -3$

(d) $\lim_{x\to 1}(1+\ln x)^{1/(x-1)}$ $= \lim_{x\to 1} (1+\ln x)^{\frac{1}{x-1}} = e^1$

$1^{\pm\infty}$

let $y = (1+\ln x)^{\frac{1}{x-1}}$

$\ln y = \frac{1}{x-1}\ln(1+\ln x)$

$= \frac{\ln(1+\ln x)}{x-1}$

$\frac{0}{0}$ $\lim_{x\to 1} \frac{\ln(1+\ln x)}{x-1} \overset{L}{=} \lim_{x\to 1} \frac{\frac{1}{1+\ln x}\cdot\frac{1}{x}}{1}$

$= \lim_{x\to 1} \frac{1}{x(1+\ln x)} = 1$

5. *Use the limit definition of the derivative* to find $f'(x)$ if $f(x)=\frac{x}{x-1}$.

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{x+h}{x+h-1} - \frac{x}{x-1}\right)(x+h-1)(x-1)}{(h)\,(x+h-1)(x-1)} = \lim_{h\to 0} \frac{(x+h)(x-1) - x(x+h-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{xh} - \cancel{x} - \cancel{h}^{1} - \cancel{x^2} - \cancel{xh} + \cancel{x}}{\cancel{h}(x+\underbrace{h}_{0}-1)(x-1)} = \frac{-1}{(x-1)^2}$$

6. (a – d) Find the derivative of each function. Simplify your final answer.

(a) $y = f(x) = (\cos(2x))^{1/x}$

$$\ln y = \frac{1}{x}\ln(\cos(2x))$$

$$\frac{1}{y}y' = \frac{1}{x}\cdot\frac{1}{\cos(2x)}(-\sin(2x))\cdot 2 + \ln(\cos(2x))(-x^{-2})$$

$$= y\left[\frac{-2\sin(2x)}{x\cos(2x)} - \frac{\ln(\cos 2x)}{x^2}\right]$$

$$= (\cos(2x))^{1/x}\left[\frac{-2\sin 2x}{x\cos 2x} - \frac{\ln(\cos 2x)}{x^2}\right]$$

(b) $g(t) = \frac{4+t}{te^t}$

$$g'(t) = \frac{te^t(1) - (4+t)(te^t + e^t)}{(te^t)^2} = \frac{te^t - 4te^t - 4e^t - t^2e^t - te^t}{t^2e^{2t}}$$

$$= \frac{-\cancel{e^t}\left[t^2 + 4t + 4\right]}{\cancel{e^t}\cdot t^2e^t} = \frac{-(t+2)^2}{t^2e^t}$$

(c) $y = \tan^{-1}\left(\frac{1}{x}\right)$

$$y' = \frac{1}{1+(1/x)^2} \cdot (-x^{-2})$$

$$= \frac{\left[-1/x^2\right]x^2}{\left[1+1/x^2\right]x^2} = \frac{-1}{x^2+1}$$

(d) $x^2 - 3xy + 2y^2 = -2$ (y is a function of x)

$$2x - 3\left[x \cdot dy/dx + y(1)\right] + 4y\, dy/dx = 0$$

$$\frac{dy}{dx}(-3x+4y) - 3y + 2x = 0$$

$$\frac{dy}{dx} = \frac{3y-2x}{4y-3x}$$

7. Show that the function $f(x) = x^3 + e^x$ has exactly one real root. (Hint: you will use the Intermediate Value Theorem and the Mean Value Theorem.)

$f(x)$ is continuous on $\mathbb{R}$.

notice: $f(-1) = -1 + 1/e < 0$ and $f(1) = 1 + e > 0$

since $f(-1) < 0 < f(1)$, by IVT a root exists.

$f'(x) = 3x^2 + e^x$ so $f(x)$ is differentiable on $\mathbb{R}$

Suppose there are two roots, a and b. Then $f(a) = f(b) = 0$

By MVT $\exists$ some $x = c$ in $[a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$

However, $f'(c) = 3c^2 + e^c \neq 0$ for any $x = c$ value.

Therefore $f(x)$ has exactly one root.

$\square$

$x(1-x) \geq 0$

$x=0,\ x=1$

8. Find the absolute maximum and absolute minimum value of $f(x) = x\sqrt{x-x^2}$ on the interval $[0,1]$.

$f(x)$ is cont. on $[0,1]$

$f(0) = 0$ min.

$f(1) = 0$

$f(3/4) = 3/4\sqrt{3/4 - 9/16}$

$= 3/4\sqrt{\frac{12-9}{16}}$

$= \frac{3\sqrt{3}}{16}$ max

$f'(x) = x \cdot 1/2(x-x^2)^{-1/2}(1-2x) + \sqrt{x-x^2}$

$= \frac{x(1-2x)}{2\sqrt{x-x^2}} + \sqrt{x-x^2} = \frac{x-2x^2+2x-2x^2}{2\sqrt{x-x^2}}$

$= \frac{3x-4x^2}{2\sqrt{x-x^2}} = 0$ when $3x-4x^2 = x(3-4x) = 0$

$x=0$ and $x = 3/4$

9. Use a linear approximation or differentials to estimate the value of $\tan\left(\frac{\pi}{4}+.02\right)$.

Linear Approx:

Let $f(x) = \tan(\pi/4 + x)$, then $f(0) = \tan \pi/4 = 1$ (point is $(0,1)$)

$f'(x) = \sec^2(\pi/4 + x)$ so $m = f'(0) = \sec^2(\pi/4) = \frac{1}{\cos^2 \pi/4} = \frac{1}{(\sqrt{2}/2)^2} = 2$

$y-1 = 2(x-0)$ or $L(x) = 2x+1$. $f(.02) \approx L(.02) = 2(.02)+1 = 1.04$

Differentials: Let $y = \tan(\pi/4 + x)$

$dy = \sec^2(\pi/4 + x)\,dx$, at $x=0$ and $dx = .02$

$dy = (2)(.02) = .04$

So, when x goes from 0 to .02, y goes from 1 to $1+.04 = 1.04$

10. An unmitigated breakout of an unknown virus initially infects one person, but rapidly spreads, infecting 9 people in 5 days. It is assumed that the number of infections grows at a rate proportional to the population of infected people. Let $p(t)$ denote the number of infections after t days.

(3) (a) Write the differential equation that $p(t)$ satisfies.

$\frac{dP}{dt} = kP$

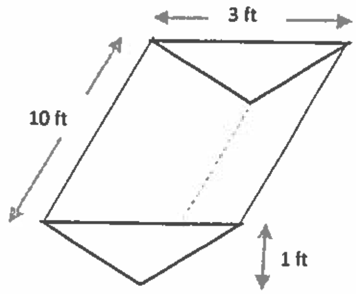
(5) (b) Determine the relative growth rate of the virus.

$P_0 = 1$ and $P(5) = 9$, solution is $P(t) = P_0 e^{kt}$

$9 = 1e^{5k}$

$\ln 9 = 5k$ so relative growth rate, $k = \ln 9/5$

11. A trough is 10 *ft* long and its ends have the shape of isosceles triangles that are 3 *ft* across at the top and have a height of 1 *ft*. If the trough is being filled with water at a rate of $12\ ft^3/min$, how fast is the water level changing when the water is 4 inches deep? Include proper units in your final answer.



Given: $dV/dt = 12\ ft^3/min$

Find: dh/dt when $h = 4in = 1/3\ ft$

$V = \frac{1}{2} b h \cdot 10 = 5bh$ where $b/h = 3/1 \therefore b = 3h$

$V = 5(3h)h = 15h^2$

$\frac{dV}{dt} = 30h\, dh/dt$

$12 = 30(1/3)\, dh/dt$

$12/10 = 6/5$

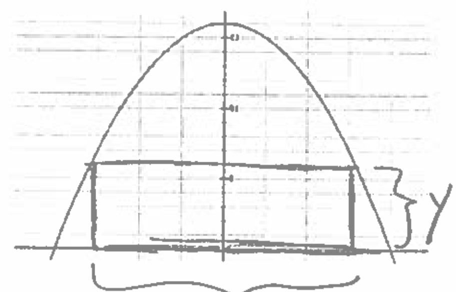
$\frac{dh}{dt} = 6/5\ ft/min$ or $1.2\ ft/min$

12. A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 16 - x^2$. What is the maximum area of such a rectangle?

Obj. function: $A = 2xy$

constraint function: $y = 16 - x^2$

$A = 2x(16 - x^2) = 32x - 2x^3$



$x \in [0, 4]$ so use closed interval method

$A' = 32 - 6x^2 = 0$

$x^2 = 32/6 = 16/3$

$x = \sqrt{16/3} = 4/\sqrt{3}$

$A(0) = 0$

$A(4) = 0$

$A(4/\sqrt{3}) = 32(4/\sqrt{3}) - 2(4/\sqrt{3})^3$

$= \frac{128}{\sqrt{3}} - \frac{128}{3\sqrt{3}}$

$= \frac{256}{3\sqrt{3}}$

13. Let $g(x) = \int_{-2007}^{x} f(t)\,dt$ where $f(t) = \int_{1}^{2t^2} \frac{\sqrt{1+u^3}}{u}\,du$. Find $g''(x)$.

$g'(x) = f(x)$ so $g''(x) = f'(x) = \frac{\sqrt{1+8t^6}}{2t^2} \cdot 4t$

$= \frac{2\sqrt{1+8x^6}}{x}$

14. Consider the function $f(x) = e^{2x-x^2}$. Find the following or state DNE if none exist.

(a) Interval(s) of increase $(-\infty, 1)$ decrease $(1, \infty)$

(b) Local maximum(s) $(1, e)$ minimum(s) DNE

Show supporting work here:

$$f'(x) = e^{2x-x^2}(2-2x) = 0 \text{ when } x = 1$$

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$x = 0$ $x = 2$

15. Consider the function $f(x) = x^4 - 4x^3 + 10$. Find the following or state DNE if none exist.

(a) Interval(s) where f is concave up $(-\infty, 0), (2, \infty)$ concave down $(0, 2)$

(b) Inflection point(s) $(0, 10)$ and $(2, -6)$

Show supporting work here:

$$f'(x) = 4x^3 - 12x^2 \quad \text{and} \quad f''(x) = 12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$$x = 0, \; x = 2$$

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0 2

$x = -1$ $x = 1$ $x = 3$

$16 - 32 + 10$

$-16 + 10 = -6$

16. Find the general antiderivative F of the function $f(x) = 5^x - \frac{3}{x}$.

$$F(x) = \frac{5^x}{\ln 5} - 3\ln|x| + C$$

17. A car is traveling at $\frac{220}{3}\ ft/s$ when the brakes are fully applied, producing a constant deceleration of $22\ ft/s^2$. What is the distance traveled before the car comes to a stop?

$v(0) = 220/3$

$a(t) = 22$ so, $v(t) = 22t + c_1$ $(c_1 = 220/3)$

$v(t) = -22t + 220/3$

$s(t) = -11t^2 + 220/3\, t + C_2$ $s(0) = 0$ so $C_2 = 0$

$s(t) = -11t^2 + 220/3\, t$, car is stopped when $v(t) = 0$

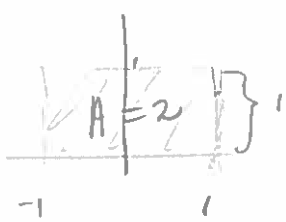
$-22t = -220/3$

$t = -\frac{220}{3}\left(\frac{1}{-22}\right) = 10/3$

$s(10/3) = -11(100/9) + 220/3\,(10/3)$

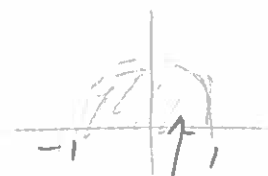
$= \frac{-1100 + 2200}{9} = 1100/9$ ft

18. Evaluate the integral, $\int_{-1}^{1} 1 - \sqrt{1-x^2}\,dx$, by interpreting it in terms of area.



$1-(-1) = 2$

$= \int_{-1}^{1} 1\,dx - \int_{-1}^{1} \sqrt{1-x^2}\,dx$



$A = \frac{1}{2}\pi(1)^2$

$= 2 - \pi/2$

19. *Use the limit definition of the definite integral* to evaluate $\int_{-2}^{0} 3x - 5\,dx$. Note: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\Delta x = \frac{0-(-2)}{n} = \frac{2}{n}$ $x_i = -2 + \frac{2}{n}i$

$\int_{-2}^{0} 3x - 5\,dx = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[3\left(-2 + \frac{2}{n}i\right) - 5\right] = \lim_{n\to\infty} \frac{2}{n}\left[\sum_{i=1}^{n} -6 + \frac{6}{n}i - 5\right]$

$= \lim_{n\to\infty} \frac{2}{n}\left[\sum_{i=1}^{n} -11 + \frac{6}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{2}{n}(-11n) + \frac{12}{n^2}\cdot\frac{n(n+1)}{2}$

$= -22 + 6 = -16$

check: $\frac{3}{2}x^2 - 5x \Big|_{-2}^{0}$

$0 - (6 + 10) = -16$

20. Evaluate each integral.

(a) $\int \sqrt{x}(6+5x)dx$ $= \int 6x^{1/2} + 5x^{3/2}\, dx$

$= \frac{12}{3}x^{3/2} + \frac{10}{5}x^{5/2} + C = 4x^{3/2} + 2x^{5/2} + C$

(b) $\int_1^4 \frac{\sqrt{y}-y}{y^2}dy$ $= \int_1^4 \frac{y^{1/2}}{y^2} - \frac{1}{y}\, dy = \int_1^4 y^{-3/2} - 1/y\, dy$

$= -2y^{-1/2} - \ln y \Big|_1^4 = \frac{-2}{\sqrt{4}} - \ln 4 - (-2 - \cancel{\ln(1)})$ 0

$= 2 - 1 - \ln 4 = 1 - \ln 4$

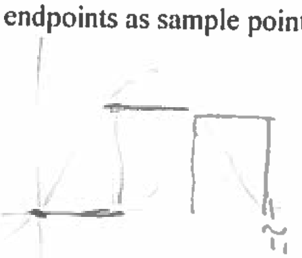
21. Write True if the statement is always true, otherwise write False and correct what is underlined.

(a) FALSE $\lim_{x\to 0^+} \ln(\sin x) = \cancel{1}$ $-\infty$

(b) FALSE Estimating the area under $f(x) = \sin x$, from $x = 0$ to $x = \pi$ using $n = 3$ rectangles and left endpoints as sample points yields the value $\cancel{\pi/3}$.

$\Delta x = \frac{\pi - 0}{3} = \pi/3$

$x_1 = 0,\ x_2 = \pi/3,\ x_3 = 2\pi/3$



$A \approx \pi/3\,[\sin(0) + \sin(\pi/3) + \sin(2\pi/3)] = \pi/3\,[0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}] = \frac{\sqrt{3}\pi}{3}$

22. The graph of f' is shown in the figure. On the same coordinate plane, sketch the graph of f if f is continuous on (0, 3) and $f(0) = -1$.