Precise Definition of a Limit

Prelab: Read definition 1 on page 83. Review Figures 3 – 6 on page 107. Read Example 2 on page 108 as well as the three paragraphs before this example.

In previous sections you were working with the “intuitive” definition of a limit. Using the “precise” definition, we can quantify how close \( x \) must be to \( a \) in order for \( f(x) \) to be within some specified distance from \( L \).

Precise Definition of a Limit: Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \). We say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

\[
\lim_{x \to a} f(x) = L
\]

if for every number \( \varepsilon > 0 \) there is a number \( \delta > 0 \) such that

To understand the definition above, a visual approach can be helpful.

Example 1: The graph of \( f(x) = x^3 + 1 \) is shown.
(a) Illustrate the above definition as it applies to the limit equation, \( \lim_{x \to 1} f(x) = 2 \).

(b) On the graph provided, label \( a \), \( L \), and \( \varepsilon \), where \( \varepsilon = 0.5 \).

(c) Calculate the value of \( \delta \) (this requires a calculator). That is, determine how close to 1 we must take \( x \) in order for \( f(x) \) to be within 0.5 of 2.
The example above shows how the precise definition of a limit is used to find a specific $\delta$, given a specific $\varepsilon$. One example is not enough to prove the limit written in 1(a). The proof of this limit must hold for any $\varepsilon$. The proof involves two parts:

1. 

2. 

Example 2: (a) Prove $\lim_{x \to 4}(1 - 2x) = -7$ using the $\varepsilon, \delta$ definition (precise definition) of a limit.

1. 

2. 

(b) Illustrate the precise definition and label $a, L, \varepsilon,$ and $\delta$. 

2-2
Work each problem showing all supporting work. You may use your textbook, lab and notes. Students may work cooperatively but each submits his/her own set of Lab Exercises.

1. (a) Use the graph below to estimate the following:

\[ \lim_{x \to 2} f(x) = \]  
\[ \delta = \text{_____ when } \varepsilon = 1 \]

(b) Label \( a, L, \varepsilon \) and \( \delta \) on the graph as in Exercises 1 and 2.

2. Prove \( \lim_{x \to 2} \left( 7 - \frac{1}{2}x \right) = 6 \) using the \( \varepsilon, \delta \) definition (precise definition) of a limit.
3. (a) The formal limit definition, “for every $\varepsilon > 0$, there exists a $\delta > 0$ such that,

$$\left| \sqrt{13 - x} - 2 \right| < \varepsilon \text{ whenever } |x - 9| < \delta$$

defines the limit equation________________________.

(b) Find $\delta$, when $\varepsilon = 1$. Show the steps of computation below.

(c) Illustrate the precise definition on the graph of $f(x)$ below and label the symbol and value for $a, L, \varepsilon$, and $\delta$. 

![Graph of a function](image-url)