Stochastic Processes and Derivative Pricing

A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Science degree in Physics from the College of William and Mary

by

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Abstract

A technique for the numerical solution of the Black-Scholes equation is presented, and this technique is extended to new solutions of the Black-Scholes equation in the presence of a deterministic time-dependent volatility term. It is shown that solutions depart significantly from the constant volatility solution around the derivative′s strike price. A new equation for valuing derivatives in the presence of stochastically varying volatility is derived and, in its solution, it is shown that investors buying put options priced using the constant volatility formula are not adequately compensated for the risk which they bear.

1. Introduction

In this project, methods in mathematical and computational physics are used in determining fair market values of derivative security instruments. This includes an analysis of stochastic processes used in the modeling of investment returns, and presents a stochastic model for the evolution of stock market volatility. First, a derivation of the Black-Scholes equation, first published in 1973 by physicist Fischer Black and economist Myron Scholes, is presented. A numerical solution to this differential equation is developed using the finite-difference method. The advantage of a numerical solution to this equation is that it allows for deterministic time dependence to easily be added to the equation′s volatility term. The effects of deterministic time-varying market volatility on the pricing of derivatives are here examined. In addition a stochastic mean-reverting volatility model is presented in an attempt to explain variances in market returns. The derivation of a new equation to value a derivative in the presence of stochastic volatility is then presented.

2. The Black-Scholes Equation

Consider a Brownian motion described by W(t), a function with the property that its differential, dW_t, is a normally distributed random variable with mean zero and variance dt, i.e. $dW_t = N(0, dt) = \sqrt{dt}N(0, 1)$, where N(a,b) is a normal distribution function with mean a and variance b. Assume a stock price, S_t, follows geometric Brownian motion with respect to time, such that:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where μ is some constant drift term and σ is some constant volatility (standard deviation) term. A security, or asset, with value $V_t(S_t, t)$ derives its value from the value S_t of some underlying stock, and is referred to as a derivative. In standard calculus, one would apply the chain rule to determine the value of dV_t. However, because S_t is a function of a random variable, Riemann calculus does not apply, and one must use stochastic calculus. Ito′s Lemma provides a generalization of the chain rule to a stochastic process of the form of equation 1. Ito′s Lemma states that dV_t is given by:

$$dV_t = \left(\mu S_t \frac{\partial V_t}{\partial S_t} + \frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial V_t}{\partial S_t} dW_t \tag{2}$$

Now, construct a portfolio wherein one sells a derivative security (initiates a short position), and buys Θ shares of its underlying stock (initiates a long position). The value of such a portfolio, X_t, is simply $\Theta S_t - V_t$, or in differential form:

$$dX_t = \Theta dS_t - dV_t \tag{3}$$

Substituting equations 1 and 2 into 3 yields:

$$dX_t = \Theta \mu S_t dt + \Theta \sigma S_t dW_t - \mu S_t \frac{\partial V_t}{\partial S_t} dt - \frac{\partial V_t}{\partial t} dt - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} dt - \sigma S_t \frac{\partial V_t}{\partial S_t} dW_t \tag{4}$$

If one chooses $\Theta = \partial V_t/\partial S_t$, then equation 4 becomes:

$$dX_t = -\frac{\partial V_t}{\partial t}dt - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2}dt = \left(-\frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2}\right) dt \tag{5}$$

The stochastic term dW_t has now dropped from the equation, and the value of the portfolio, X_t, now evolves according to an entirely deterministic process. Investment in such a portfolio would involve no risks; the outcome is certain. Under the no arbitrage assumption (assumption of no guaranteed riskless profits), the value of such an investment must provide the same return as an investment in risk free bonds which provide an interest rate r. The value of a risk free bond portfolio, Y_t, is $Y_0 e^{rt}$. Differentiation with respect to t yields:

$$dY_t = rY_0 e^{rt} dt = rY_t dt \tag{6}$$

Because X_t and Y_t both involve no risk, their differentials must be of the same form; $dX_t = rX_t dt$. Equating equations 5 and 6 yields:

$$\left(-\frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2}\right) dt = r\left(S_t \frac{\partial V_t}{\partial S_t} - V_t\right) dt \tag{7}$$

Which reduces to:

$$\frac{\partial V_t}{\partial t} + rS_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} = rV_t \tag{8}$$

Equation 2.8 is the famous Black-Scholes equation, published by Fischer Black and Myron Scholes in 1973 [1]. An analytic solution to this equation can be determined by making the substitutions $V_t = e^{rt}u, y = ln(S_t), \tau = T - t, andz = y(r - 0.5\sigma^2)\tau$ where T is the time of the derivatives expiration. With these substitutions, equation 8 becomes:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial z^2} \tag{9}$$

This is simply a one dimensional diffusion equation, for which the Greens function is known to be:

$$G(z,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{z^2}{2\sigma^2\tau}\right) \tag{10}$$

One can obtain an analytic solution through taking an inner product of equation 10 and a given initial condition, $u_0(z)$ to arrive at:

$$u(z,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int \exp\left(-\frac{(z-y)^2}{2\sigma^2\tau}\right) u_0(y)dy \tag{11}$$

3. Numerical Solution to Black Scholes

When one introduces deterministic time dependence to the volatility term, the substitution of variables no longer produces a diffusion equation, and the solution of the Black-Scholes equation becomes more difficult. For this reason, a numerical solution was implemented using the finite difference method as outlined in Numerical Recipes [2]. The derivatives in the Black Scholes equation were discretized as follows:

$$\frac{\partial V_t}{\partial S_t} = \frac{V_{i,j+1} - V_{i,j-1}}{2(\Delta S)}$$

$$\frac{\partial^2 V_t}{\partial S_t^2} = \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta S)^2}$$

$$\frac{\partial V_t}{\partial t} = \frac{V_{ij} - V_{i-1,j}}{\Delta t}$$

Where the indices i and j refer to discrete time and stock price values respectively, and Δt and ΔS are the selected time and stock price step sizes. In order for solutions to converge, one should choose $\Delta t << \Delta S$. Substituting the discretized derivatives into equation 8 and rearranging terms yields:

$$V_{i-1,j} = \Delta t \left(\frac{1}{2}\sigma^2 \frac{S_i^2}{(\Delta S)^2} - \frac{1}{2} r \frac{S_i}{\Delta S}\right) V_{i,j-1} + \left[1 - \Delta t \left(r + \sigma^2 \frac{S_i^2}{(\Delta S)^2}\right)\right] V_{i,j}$$

$$+\Delta t \left(\frac{1}{2}\sigma^2 \frac{S_i^2}{(\Delta S)^2} + \frac{1}{2} r \frac{S_i}{\Delta S}\right) V_{i,j+1} \tag{12}$$

Because the value of the derivative at expiration time T is the known boundary condition, this value can be substituted in for $V_{T,j}$ for all stock prices j. Equation 12 can then be used to calculate the value of the derivative one time

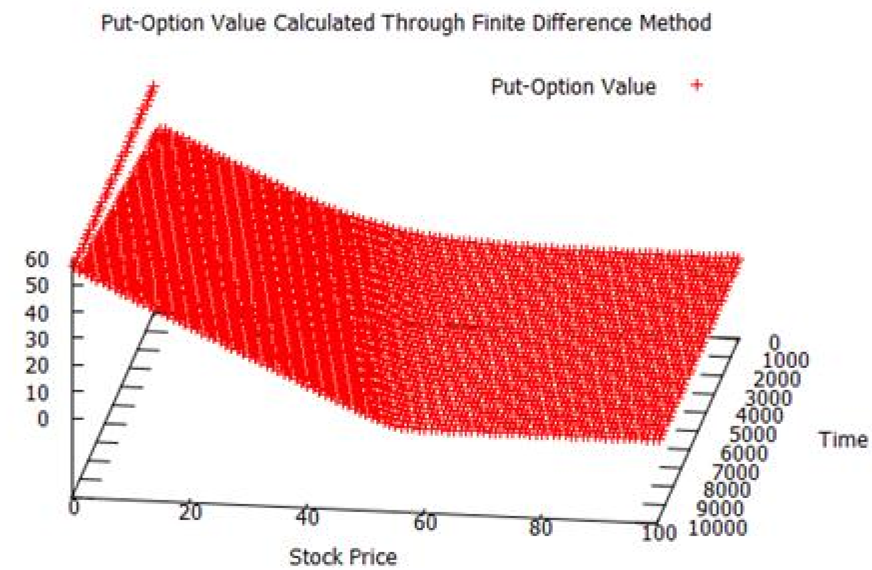

Figure 1: Put-Option values for K = 55 price units, r = 0.03 per time unit, σ^2 = 0.01 per time unit, T = 10 time units, Δt = 0.001 time units, ΔS = 1 price unit.

step before T. This new value is then substituted into the right hand side of equation 12, and equation 12 can once again be used to find the value of the derivative two time steps before T. This process is repeated until one arrives at time t = 0.

Using this numerical algorithm, the prices of a European put-option were calculated over a mesh of time and stock price values. A European put-option gives the owner the right, but not the obligation, to sell a share of some underlying stock for a strike price, K, regardless of the market price S. Such an option may only be exercised at the expiration time T. The boundary condition for this option is:

$$V_{T,j} = max[K - S_j, 0] \tag{13}$$

The finite-difference method was used to calculate the values for a European put-option with strike price 55. The results are displayed in a plot in Figure 1.

4. Monte Carlo Simulation of a Stock Market

In order to demonstrate the equivalence of risk free assets, a simulation of a stock market was constructed. In this simulation, it is assumed that there exists only one stock, and that there are no transaction costs. It is also assumed that an investor is only able to buy or sell shares of stock once during a given time interval. The stock price in this simulation follows the geometric Brownian motion process described by equation 1, with the following parameters:

$$S_0 = 50$$

$$\mu = 0.1$$

$$\sigma^2 = 0.2$$

$$K = 55$$

$$T = 300$$

The investor in this simulation begins by selling a call-option at the first time step at the Black-Scholes price. This call-option gives the investor the right, but not the obligation, to purchase shares of an underlying stock at a strike price, K. He then buys a number of stock units equivalent to the derivative of the call option value with repect to the underlying stock price. At each time step, the investor adjusts his stock position so that the number of shares in his possession remains equal to the value of the derivative of the option value with respect to the stock value. At each time step, the net wealth of the investor is measured. The investors wealth was measured for a total of 300 time units for 400 independent trials. The results of each trial were averaged together.

In figure 2, a sample path of the investors wealth and the average path are displayed. It can be seen that, on average, the investors wealth may be fit to an exponential growth function with an R^2 of 0.9867. The investors portfolio in

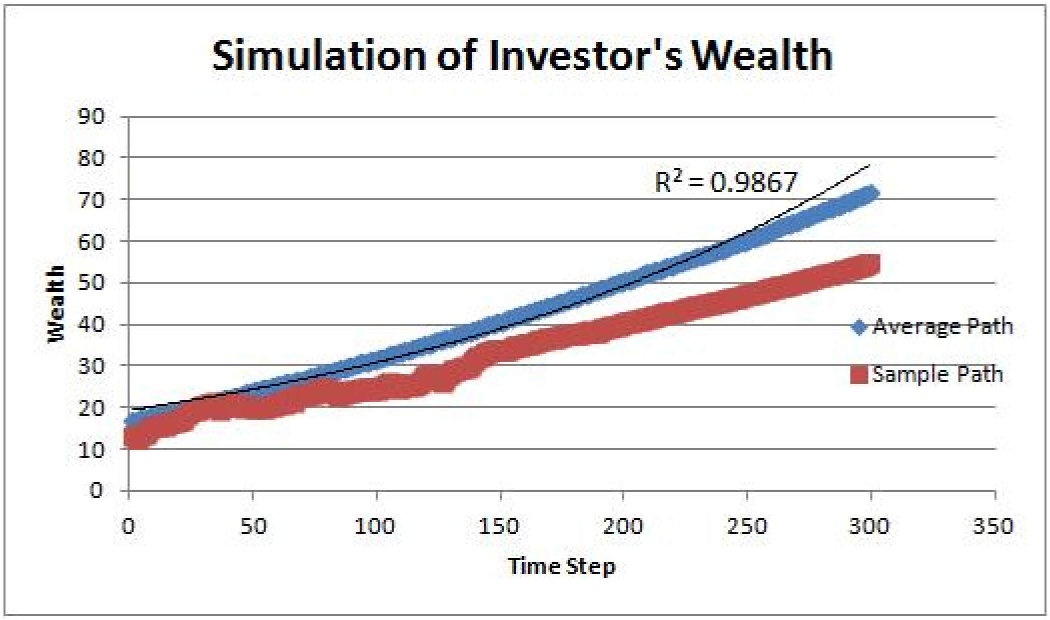

Figure 2: Wealth of an investor selling calls and purchasing underlying stock.

this simulation is therefore, on average, follows the exponential growth predicted by equation 6. This portfolio equivalence is built into the assumptions of the Black-Scholes equation.

5. Deterministic Volatility Time Dependence

An advantage of the numerical solution produced is that time dependence can quite easily be added to the volatility term. One simply must modify the volatility term in equation 8 to be dependent on the time index, i, and then recalculate the derivative values. Values for a European put-option were calculated for three differenct cases of volatility: a sinusoidal volatility term, a linear volatility term, and a step volatility term. The specific functions were:

$$\sigma_{sin}(i) = 0.2\sqrt{2}sin^2(i/35)$$

$$\sigma_{linear}(i) = 0.00001i + 0.05$$

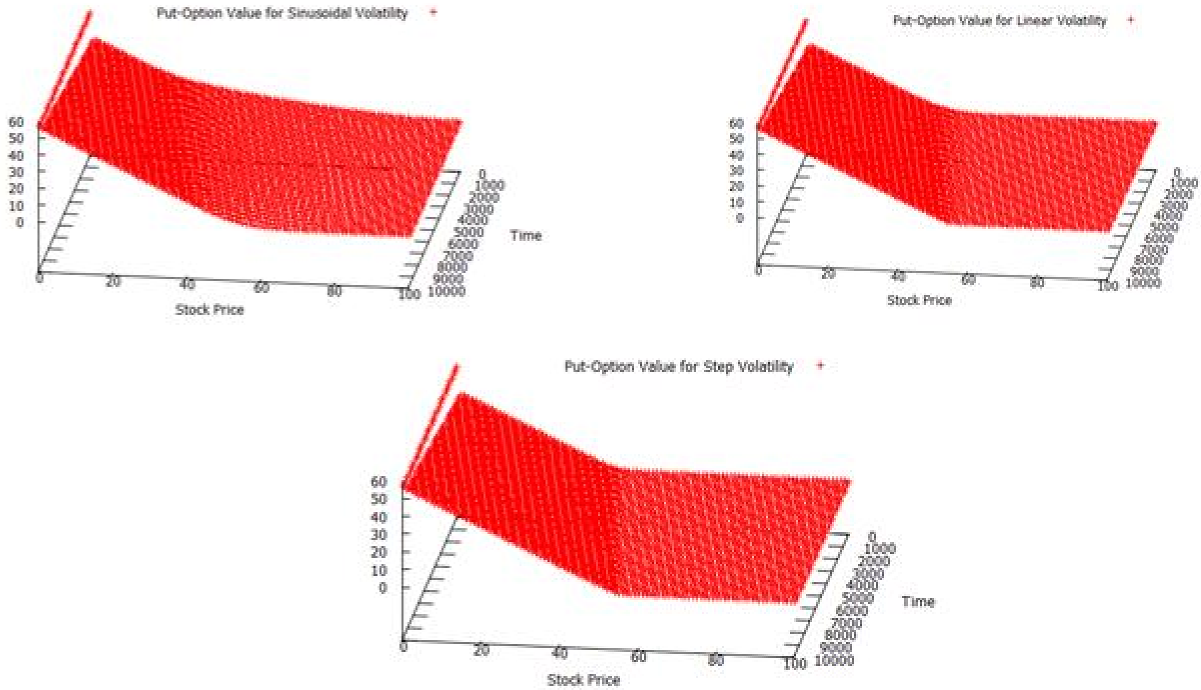

Figure 3: Put-Option values for sinusoidal, linear, and step volatility dependences.

$$\sigma_{step}(i) = \{ 0 .025, i < 5000; 0.4, i \geq 5000$$

These three cases were examined because each can reflect volatility conditions under different time scales. A sinusoidal function was used because market volatility tends to exhibit oscillatory behavior. A linear function was used because, in the intermediate term, market volatility can at times be observed to increase or decrease steadily. The step function was used to reflect the short-term jumps in volatility that tend to occur. Plots of put-option values for time dependent volatilities are displayed in Figure 3.

In Figure 4, the differences between the constant volatility and the time varying volatility solutions are displayed.

It can be seen from Figure 4 that time dependent volatilities dont strongly

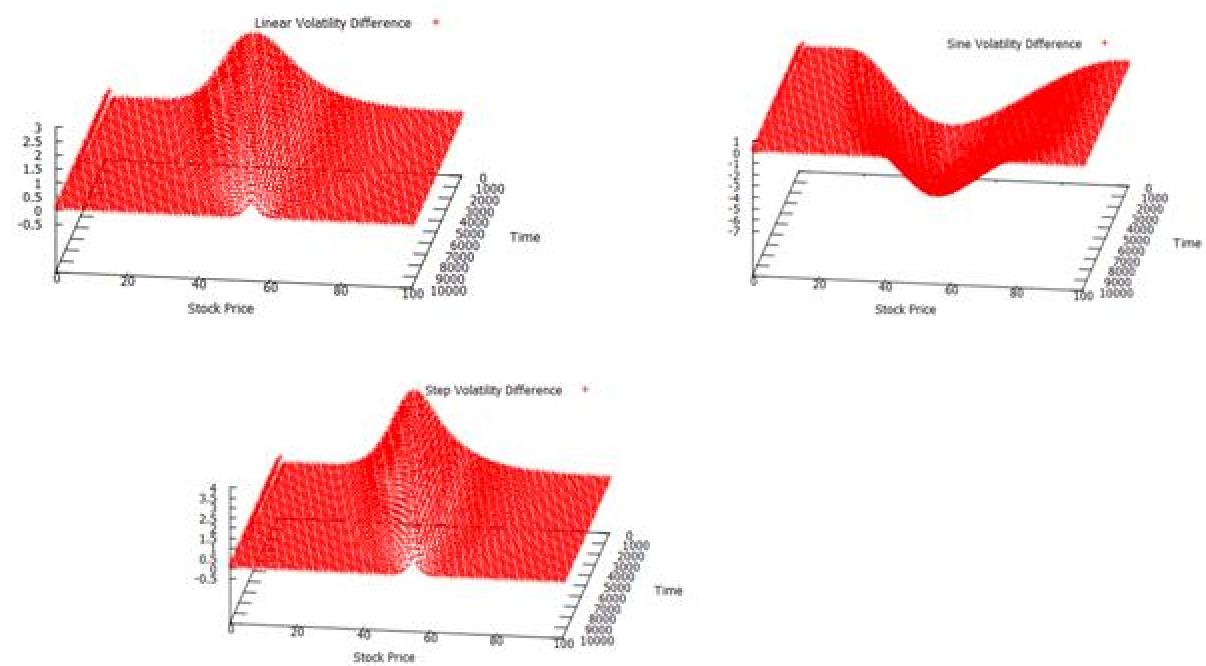

Figure 4: Differences between the constant volatility solution and time-dependent volatility solutions.

affect derivative prices at stock price values far from the strike price. This is reasonable, as the most drastic change in the derivative price occurs when one steps across the strike price. When volatility is higher, the probability of crossing the strike price is higher, and when volatility is lower, the probability of crossing the strike price is lower. Time varying volatility will therefore cause large changes in the desirability of holding a derivative security, which causes the difference in pricing.

6. Stochastic Volatility Time Dependence

One can use macroeconomic factors to predict changes in volatility and attempt to fit these changes to a deterministic model; however any deterministic models will only be valid for a short time frame. Changes in market volatility can best be characterized by a stochastic process. Figure 5 shows a plot of the standard deviations of the 21 day returns of the SPY ETF (a fund which seeks to match the performance of the S&P 500 Index) from February of 1993 through September of 2012, as well as the autocorrelation function for the series. The value of the autocorrelation function, ρ_l at each lag, l, is given by:

$$\rho_l = \frac{Cov(\sigma_t, \sigma_{t-l})}{Var(\sigma_t)} \tag{14}$$

The oscillatory autocorrelation function seems to indicate that volatility tends to oscillate about some mean value. From the plot of the volatility itself, reversion to some mean value is also visible. This indicates that market volatility could likely be fit to a stochastic mean reverting process: a process which exhibits stochastic fluctuations as well as a deterministic reversion to a mean.

The model used to describe this volatility is simply the Cox, Ingersoll, Ross (CIR) mean reverting model for interest rates[3]. According to this model, the volatility is described by:

$$d\sigma = \theta(\mu - \sigma)dt + \nu\sqrt{\sigma}dB \tag{15}$$

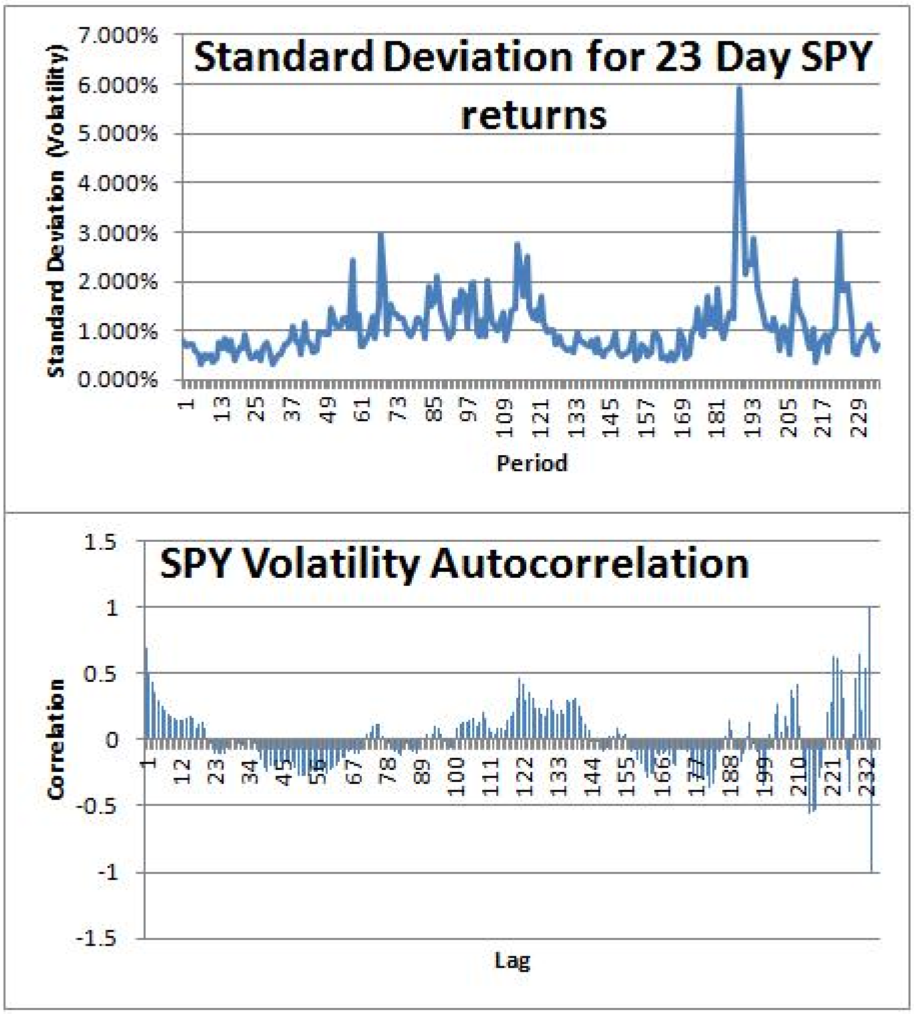

Figure 5: Market volatility as calculated through standard deviations of the SPY return, and the accompanying autocorrelation function.

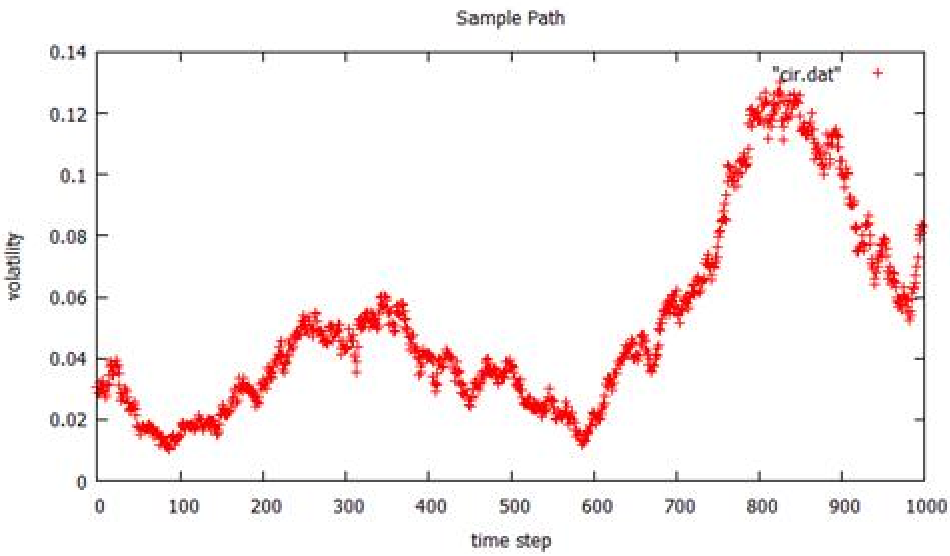

Figure 6: Sample Path for a CIR process with $\theta = 3$, $\mu = 0.02$, $\nu = 0.01$, and $\sigma_0 = 0.03$

The first term in this equation results in a deterministic return to the mean, while the second term results in stochastic deviations from, or reversions to, the mean. The θ term is the logarithmic velocity of reversion to the mean, μ, while dB is a Brownian Motion term, and ν is a standard deviation.

A sample path for this process is displayed in Figure 6. Sample paths generated according to this process tend to be less noisy than the SPY volatility, and also lack the dramatic spiking behavior, indicating that the model must be improved.

7. Black Scholes with Stochastic Volatility

Starting with the volatility model presented in equation 15, and assuming the price of a stock is an unknown function of this volatility, a Taylor expansion yields:

$$dS = \frac{\partial S}{\partial \sigma} d\sigma + \frac{\partial S}{\partial t} dt + \frac{1}{2}\frac{\partial^2 S}{\partial \sigma^2}(d\sigma)^2 + \frac{\partial^2 S}{\partial \sigma \partial t} d\sigma dt + \frac{1}{2}\frac{\partial^2 S}{\partial t^2}(dt)^2 + ... \tag{16}$$

Squaring equation 15 for d gives:

$$(d\sigma)^2 = \theta(\mu - \sigma)dt^2 + \nu^2\sigma(dB)^2 + 2\theta\nu(\mu - \sigma)\sqrt{\sigma}dBdt \tag{17}$$

Because dB is of order $\sqrt{dt}$, the dBdt term can be omitted, along with the dt^2 term. It can be shown that the expectation value of dB^2 is dt, and its variance is of order dt^2. The dB^2 term is therefore approximately equal to dt. Making this substitution, omitting higher order terms, and inserting equation 17 into 16 yields:

$$dS = \left[\frac{\partial S}{\partial \sigma}\theta(\mu - \sigma) + \frac{\partial S}{\partial t} + \frac{1}{2}\nu^2\sigma\frac{\partial^2 S}{\partial \sigma^2}\right] dt + \nu\sqrt{\sigma}\frac{\partial S}{\partial \sigma}dB \tag{18}$$

Now we assume the value of a derivative, V, is a function of this stock price, and Taylor expand to obtain:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial^2 V}{\partial S \partial t}dSdt + ... \tag{19}$$

Where the dS^2 term is given by:

$$(dS)^2 = \nu^2\sigma(\frac{\partial S}{\partial \sigma})^2(dB)^2 = \nu^2\sigma(\frac{\partial S}{\partial \sigma})^2 dt \tag{20}$$

Inserting equation 20 into equation 19 and omitting higher order terms gives:

$$dV = \left[\frac{\partial V}{\partial S}\frac{\partial S}{\partial \sigma}\theta(\mu - \sigma) + \frac{\partial V}{\partial S}\frac{\partial S}{\partial t} + \frac{\partial V}{\partial t} + \frac{1}{2}\nu^2\sigma\left(\frac{\partial V}{\partial S}\frac{\partial^2 S}{\partial \sigma^2} + \frac{\partial^2 V}{\partial S^2}(\frac{\partial S}{\partial \sigma})^2\right)\right] dt + \nu\sqrt{\sigma}\frac{\partial V}{\partial S}\frac{\partial S}{\partial \sigma}dB \tag{21}$$

We now construct the exact same portfolio considered in section two, wherein we buy a number of shares of stock equal to the derivative of the derivative security′s value with respect to the underlying stock, and sell a derivative security. The differential value of this portfolio is given by:

$$dX = \frac{\partial V}{\partial S}dS - dV \tag{22}$$

Inserting in the previously determined equations for dS and dV gives:

$$dX = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\nu^2\sigma\frac{\partial^2 V}{\partial S^2}\left(\frac{\partial S}{\partial \sigma}\right)^2\right) dt \tag{23}$$

Once again, the stochastic term cancels, so the portfolio must evolve exponentially at the risk free rate. Equating equation 22 with the differential value of a risk free portfolio as given in equation 6 yields:

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\nu^2\sigma\frac{\partial^2 V}{\partial S^2}\left(\frac{\partial S}{\partial \sigma}\right)^2\right)dt = r\left(S\frac{\partial V}{\partial S} - V\right)dt \tag{24}$$

Cancelling terms gives:

$$rV = \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\nu^2\sigma\frac{\partial^2 V}{\partial S^2}\left(\frac{\partial S}{\partial \sigma}\right)^2 \tag{25}$$

Equation 25 is a deterministic partial differential equation describing the value of a derivative security for which the underlying asset is subject to a volatility described by a CIR process.

A numerical solution to this equation was constructed using the same method used in section 3. A plot of the solution, along with the difference between the constant volatility solution and the stochastic volatility solution is presented in Figure 7. The solution shows the value of the derivative along a path of fixed volatility. It can be seen that the stochastic volatility solution predicts a smaller value for put options than the constant volatility solution. This would mean that investors typically overpay for these put options, and that the fair value may be slightly less than the market value. This is reasonable, as an asset with varying volatility is a riskier asset than an asset with fixed volatility. An investor in a riskier asset would require a higher expected return, and would therefore place less value on the riskier asset. Because an option subject to stochastic volatility is more risky than a constant volatility option, the option with stochastic volatility will not be priced as high.

8. Conclusion

Methods for the pricing of options under the more realistic assumption of time varying volatility have been presented. A method for inserting deterministic volatility time dependence into the Black-Scholes equation is presented, and shows that the path along which the volatility evolves in time can cause

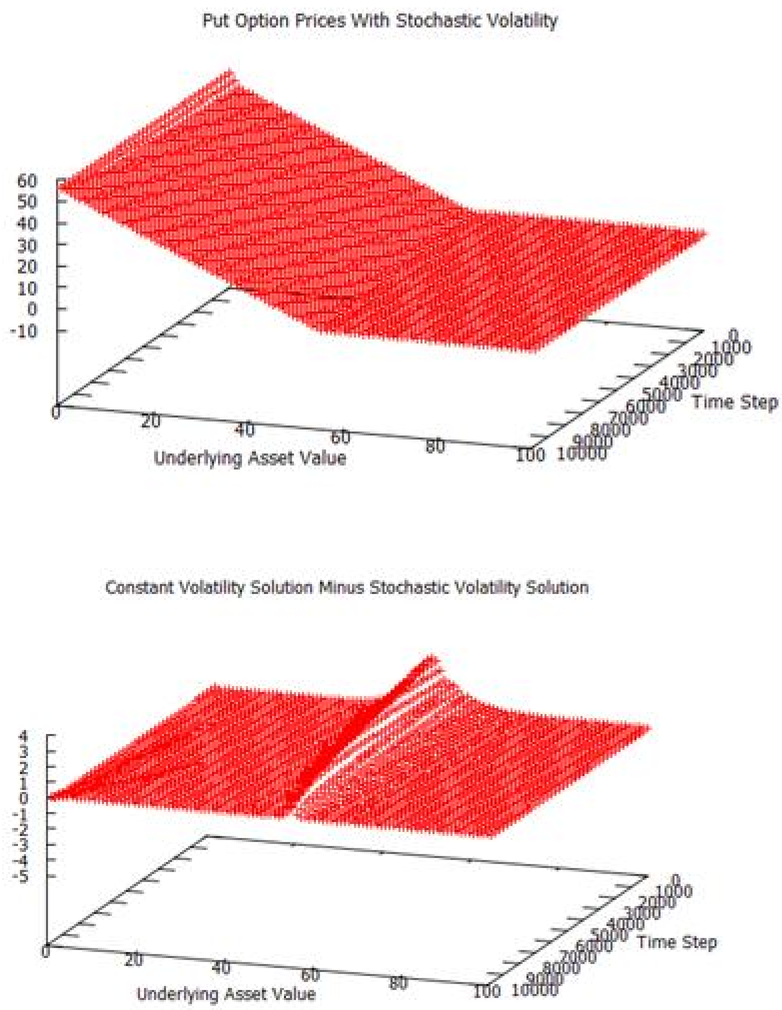

Figure 7: Stochastic volatility put-option value, and its difference from constant volatility value, along the solution with σ =0.1 at each point, K = 55 price units, r = 0.03 per time unit, ν = 0.001.

changes in a derivatives fair value. Typically institutional investors will forecast volatility conditions at a derivatives expiration, and use this volatility forecast to price the derivative. However, it has been shown that in order to obtain a more fair value for the derivative, investors must also forecast the path which the volatility will follow to reach its final value.

The effects of a stochastic mean reverting model for volatility have also been shown to cause a change in the fair price of a derivative. Using the principle that there can be no guaranteed riskless profits, a formula has been derived to value a derivative on an asset subject to such a stochastic mean reverting volatility. The solution of this formula for a put option shows that investors typically over pay for put options, as they fail to take into account the risk that volatility will change.

9. References

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