Cosmological Implications of Stochastic Quantum Gravity

A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Science degree in Physics from the College of William and Mary

by

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Abstract

We show that a stochastic description of quantum field theory may explain the anomalously low quadrupole moment in the temperature power spectrum of the cosmic microwave background. Stochastic quantum field theory permits non-equilibrium states and features the Born rule as an equilibrium description of probability distributions. We study the stochastic harmonic oscillator, which converges to the quantum harmonic oscillator at a rate that is inverse exponential in the frequency ω. We then describe a simple inflaton field as a mode expansion of stochastic harmonic oscillators. In this model, the modes of the inflaton field have smaller widths than expected in quantum field theory when they are frozen by inflation. We suggest that this may cause a decrease in low-ℓ moments of the temperature power spectrum.

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1 Introduction

1.1 The Quadrupole Anomaly

The cosmic microwave background (CMB) is a natural place to look for signs of beyond-Standard Model physics. Small changes in the quantum behavior of fields are writ large by inflation, preserving a record of small-scale, short-time phenomena that would normally be nearly unmeasurable. As a result, it is a goal of many beyond-Standard Model theories to explain any anomolies in CMB data.

One such deviation is the anomalously low quadrupole moment in the CMB temperature power spectrum. Both the Planck mission [1] and the Wilkinson Microwave Anisotropy Probe [2] have measured this anomaly, which was noted by the Planck mission to deviate from the expected value by greater than 1σ even once cosmic variance is factored in [3]. The fact that this discrepancy appears in multiple measurements suggests that the anomaly is no more artifact of measurement. Thus the cause of the quadrupole anomaly is largely attributed to one of two causes: either new physics or greater cosmic variance.

Cosmic variance is essentially the observation that we only have one sky to measure. Since inflation cannot be reproduced in a lab, measurements of the CMB constitute one irreproducible measurement of a probability distribution. From this view, the presence of a few anomalies should not be a surprise. This argument particularly applies to the low-ℓ domain which our chosen anomaly lies in. Most theories assume that the spherical harmonic coefficients $a_{\ell m}$ are drawn from the same distribution for each ℓ, giving us $2\ell + 1$ measurements of the 2^ℓ-pole distribution. Thus the low-ℓ are drawn from a smaller sample size, further increasing the likelihood of anomalies. However, even with this uncertainty factored in, the quadrupole moment of the temperature power spectrum is still unexpectedly low. In order to explore this persistent anomaly, and to sate our curiosity, we therefore choose to search for a possible physical origin of this discrepancy.

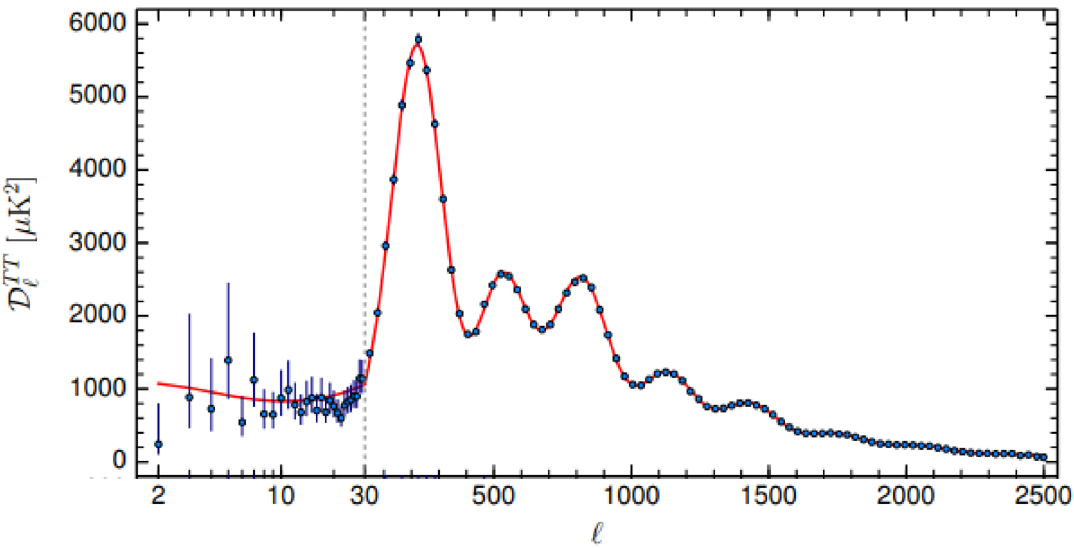

Figure 1.1: Planck measurements of the multipole coefficients of CMB temperature isotropy vs. theoretical predictions of ΛCDM. Error bars are 1σ, including cosmic variance. Note the anamalously low $\ell = 2$ moment. [1]

1.2 Non-Equilibrium Deviations from Quantum Theory

One category of theories that have been investigated for explaining the low-ℓ temperature anomaly is non-equilibrium quantum theory. Such theories feature standard rules of quantum mechanics, such as the Born rule, not as axioms but rather dynamic equilibria that a system approaches over time. Perhaps the best known example of this is de Broglie-Bohm theory, in which an arbitrary ensemble of particles is guided over time by a pilot wave into a distribution that agrees with the Born rule. de Broglie-Bohm theory was first derived in an attempt to give a "realist" interpretation of quantum theory. In recent years, it has been used by some (notably Valentini [4]) to explain anomalies in the CMB. While we will not be investigating these theories here, they provide a useful blueprint for how a non-equilibrium theory can lead to discrepancies in the temperature power spectrum.

Instead, we will suggest how a newer non-equilibrium theory of stochastic quantum gravity may explain these anomalies. Like de Broglie-Bohm theory, this theory was not originally developed for the purpose of explaining CMB anomalies. Rather, it was motivated by the fact that a discrete Poisson scattering of stochastic events in spacetime might provide a physical ultraviolet regulator for use in Sakharov's procedure for induced quantum gravity [5] [6]. Furthermore, on a large scale, these discrete kicks will appear continuous, causing functions of spacetime to evolve with stochastic differential equations. By following the process of Nelson, quantum mechanics then emerges from stochastic mechanics [7]. These stochastic differential equations (SDEs) give rise to the non-equilibrium nature of the theory. Arbitrary ensembles of particles or field densities are allowed as initial conditions, but over time they will tend towards an equilibrium distribution that agrees with quantum theory.

In this paper, we will use this non-equilibrium behavior to suggest an explanation for the low-ℓ temperature anomaly. First, we will develop the basic theory of stochastic mechanics and describe the stochastic harmonic oscillator. Next, we will review CMB anomalies and how they arise from inflationary cosmology. Finally, we will describe a version of the standard inflation field using stochastic harmonic oscillators and suggest how the slow evolution of its low-k modes could produce the low-ℓ anomaly. We will provide avenues of future research on this topic, such as the derivation of the Schrödinger equation from stochastic mechanics in expanding spacetime.

2 Stochastic Mechanics

The basis of most stochastic quantum theories is Nelson's work on stochastic mechanics. Nelson showed that under basic assumptions, the Schrödinger equation can be derived from the evolution of a stochastic process. We will show a version of this derivation here that largely follows de la Peña, Cetto, and Hernández [8]. We will then discuss how stochastic states obey the Born rule as an equilibrium, rather than a postulate; as an example, we will consider the stochastic harmonic oscillator, which will play a key role in stochastic quantum field theory.

2.1 Mechanics 1: Stochastic Derivatives

In order to do dynamics in a stochastic system, we first need to define derivatives. We will start with the fundamental differential equation for a stochastic process x:

$$dx = (v + u)dt + \sqrt{2D}dW \tag{2.1}$$

where v and u are the **convective and stochastic velocities** and D is the **diffusion coefficient**. This contains the standard variance with respect to time, along with a term dW proportional to the **Wiener process**. Broadly speaking, this term produces a "kick" with a normally distributed direction and magnitude at each timestep. It is characterized by the relation $dW^2 = dt$; this causes many expansions of differentials to include quadratic terms which are ignored in normal calculus.

In stochastic dynamics, we consider derivatives of deterministic functions $f(x)$ on stochastic variables x. These are calculated in the usual way using the chain rule. However, this is complicated because the differential dx of a stochastic variable includes a term proportional to $dW \sim dt^{1/2}$. Thus the terms in the chain rule proportional to dx^2 will have elements of order dt. Taking this into account yields two independent definitions of the derivative of $f(x)$, commonly called the **convective and stochastic derivatives**.

We will start with the convective derivative. This is derived by considering a standard expression for the derivative of $f(x)$, defining the forward and backward differences

$\Delta_+(x) = x(t+\Delta t) - x(t)$ and $\Delta_-(x) = x(t) - x(t-\Delta t)$ for brevity:

$$\frac{f(\Delta_+(x)) - f(\Delta_-(x))}{2\Delta t} \tag{2.2}$$

Expanding this with the chain rule gives

$$\frac{f(\Delta_+(x)) - f(\Delta_-(x))}{2\Delta t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}\frac{\Delta_+(x_a) + \Delta_-(x_a)}{2\Delta t} + \frac{\partial^2 f}{\partial x^a dx^b}\frac{\Delta_+(x_a)\Delta_+(x_b) - \Delta_-(x_a)\Delta_-(x_b)}{4\Delta t} + \ldots \tag{2.3}$$

In normal calculus, we can take the limit $\Delta t \to 0$ and find that all terms after the second vanish. However, in this case, the differences Δt contain terms proportional to $dW \sim dt^{1/2}$, so this limit cannot be taken. We can avoid this issue by taking the expectation value of both sides. Since dW represents Gaussian noise, it vanishes on average. This gives us the expression

$$\frac{\langle f(\Delta_+(x)) - f(\Delta_-(x))\rangle}{2\Delta t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}\frac{\langle\Delta_+(x_a)\rangle + \langle\Delta_-(x_a)\rangle}{2\Delta t} \tag{2.4}$$

$$+\frac{\partial^2 f}{\partial x^a dx^b}\frac{\langle\Delta_+(x_a)\Delta_+(x_b)\rangle - \langle\Delta_-(x_a)\Delta_-(x_b)\rangle}{4\Delta t} + \ldots \tag{2.5}$$

where we have used the fact that the expectation value of a sum is the sum of the expectation values.

We can show that the second term vanishes. The only term in the products $\Delta_+(x_a)\Delta_+(x_b)$ and $\Delta_-(x_a)\Delta_-(x_b)$ that is proportional to dt is the product of the noise terms dW. But the noise is symmetric in time; the process gets the same distribution of kicks whether it is moving forward or backward in time. Thus the dt terms $\Delta_+(x_a)\Delta_+(x_b)$ and $\Delta_-(x_a)\Delta_-(x_b)$ are equal; thus to order dt, the difference $\langle\Delta_+(x_a)\Delta_+(x_b)\rangle - \langle\Delta_-(x_a)\Delta_-(x_b)\rangle$ vanishes. This leads us to the definition of the **convective derivative**, denoted as D_c:

$$D_c(f(x)) = \frac{\langle f(\Delta_+(x)) - f(\Delta_-(x))\rangle}{2\Delta t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}\frac{\langle\Delta_+(x_a)\rangle + \langle\Delta_-(x_a)\rangle}{2\Delta t} \tag{2.6}$$

The operator expression of D_c is thus

$$D_c = \frac{\partial}{\partial t} + \frac{\langle\Delta_+(x_a)\rangle + \langle\Delta_-(x_a)\rangle}{2\Delta t}\cdot\nabla$$

One might notice that this is similar to the continuity equation. We make the comparison more clear by defining the **convective velocity**

$$v = \frac{\langle \Delta_+(x_a) \rangle + \langle \Delta_-(x_a) \rangle}{2\Delta t} \tag{2.7}$$

which allows us to write the convective derivative in the compact form

$$D_c = \frac{\partial}{\partial t} + v \cdot \nabla \tag{2.8}$$

This is one way to define the derivative of a deterministic function of a stochastic variable. However, these is another, equally powerful way, and it is this definition that earns the title of **stochastic derivative**.

The idea behind the stochastic derivative is to take the *sum*, rather than the difference, of the function at nearby points:

$$\frac{\langle f(\Delta_+(x)) + f(x(t - \Delta t)) \rangle}{2\Delta t} \tag{2.9}$$

Just as before, we can expand this with the chain rule, with the difference being in the signs and the new term proportional to $\frac{f(x)}{\Delta t}$:

$$\frac{f(x(t+\Delta t)) + f(x(t-\Delta t))}{2\Delta t} = \frac{2f(x)}{2\Delta t} + \frac{\partial f}{\partial x_a}\frac{\Delta_+(x_a) - \Delta_-(x_a)}{2\Delta t} \tag{2.10}$$

$$+\frac{\partial^2 f}{\partial x_a \partial x_b}\frac{\Delta_+(x_a)\Delta_+(x_b) + \Delta_-(x_a)\Delta_-(x_b)}{2\Delta t} + .. \tag{2.11}$$

This definition by itself doesn't make much sense, since the term $\frac{2f(x)}{\Delta t}$ will blow up as $\Delta t \to 0$. To get around this, we manually subtract this term from our definition of the stochastic derivative, and take the expectation value to get rid of dW terms:

$$D_s f(x) := \frac{\langle f(x(t+\Delta t) + f(x(t-\Delta t)) - 2f(x(t)) \rangle}{2\Delta t} \tag{2.12}$$

Thus we define the **stochastic derivative** D_s as

$$D_s f(x) = \frac{\partial f}{\partial x_a}\frac{\langle \Delta_+(x_a) \rangle - \langle \Delta_-(x_a) \rangle}{2\Delta t} + \frac{\partial^2 f}{\partial x_a \partial x_b}\frac{\langle \Delta_+(x_a)\Delta_+(x_b) \rangle + \langle \Delta_-(x_a)\Delta_-(x_b) \rangle}{2\Delta t} \tag{2.13}$$

Just as before, we will redefine the terms proportional to partials of f as "velocities". The term proportional to $\frac{\partial g}{\partial x_a}$ will be called the **stochastic velocity**, as denoted as u:

$$u := \frac{\langle \Delta_+(x_a) \rangle - \langle \Delta_-(x_a) \rangle}{2\Delta t} \tag{2.14}$$

The second-order term is more complicated. Unlike in the convective derivative, the second-order terms are summed, so this term does not disappear. Furthermore, since this term has two lower indices, it cannot be defined as a vector velocity like v or u; instead, it must be defined as a $(0,2)$ tensor called the **diffusion tensor** and denoted D_{ab}:

$$D_{ab} := \frac{\langle \Delta_+(x_a)\Delta_+(x_b) \rangle + \langle \Delta_-(x_a)\Delta_-(x_b) \rangle}{2\Delta t} \tag{2.15}$$

With these definitions, we can rewrite the stochastic derivative in a continuity equation-like form:

$$D_s f(x) = u_a \cdot \frac{\partial f}{\partial x_a} + D_{ab} \frac{\partial^2 f}{\partial x_a \partial x_b} \tag{2.16}$$

In general, this is a difficult system to analyze. However, in all cases of interest to us, the diffusion tensor is diagonal and has the same element along the entire diagonal; that is, D_{ab} just behaves like a scalar. Thus we can drop the indices and simply denote D_{ab} as the **diffusion coefficient** D. This allows us to rewrite the stochastic derivative in a much simpler form:

$$D_s f(x) = u \cdot \nabla f + D\nabla^2 f \tag{2.17}$$

Removing f gives us the operator form of the stochastic derivative:

$$D_s = u \cdot \nabla + D\nabla^2 \tag{2.18}$$

2.2 Mechanics 2: The Nelson-Newton Equation

We have defined the **convective and stochastic derivatives** of a deterministic function of a stochastic variable as

$$D_c = \frac{\partial}{\partial t} + v \cdot \nabla \tag{2.19}$$

$$D_s = u \cdot \nabla + D\nabla^2 \tag{2.20}$$

Note the useful relations

$$D_c x = v \tag{2.21}$$

$$D_s x = u \tag{2.22}$$

Nelson's insight was to use these derivatives to write a stochastic equivalent of Newton's law. The force F is defined in our case as the usual gradient of a potential $-\nabla V(x)$. However, since there are two derivatives in play, there are four independent accelerations:

$$a_{cc} = D_c D_c x = D_c v \tag{2.23}$$

$$a_{cs} = D_c D_s x = D_c u \tag{2.24}$$

$$a_{sc} = D_s D_c x = D_s v \tag{2.25}$$

$$a_{ss} = D_s D_s x = D_s u \tag{2.26}$$

Thankfully, we are spared from having four different equations of motion by a symmetry argument. Since we aim to create an analogue to Newton's second law, we expect the force to be proportional to a linear combination of the accelerations; that is, we expect

$$F = m\left(\lambda_{cc} a_{cc} + \lambda_{cs} a_{cs} + \lambda sc a_{sc} + \lambda_{ss} a_{ss}\right) \tag{2.27}$$

We will only be considering forces F that are gradients of some potential $V(x)$ which depends only on position. This is useful because a function of position must be invariant under time reversal. It can be shown that the convective derivative D_c is invariant under time reversal, while D_s picks up a minus sign. Thus the accelerations a_{cc} and a_{ss} are invariant under time reversal, while a_{cs} and a_{sc} pick up the minus sign of their lonely D_s. But this means that, in order for F to be invariant under time reversal, we must have $\lambda_{cs} = \lambda_{sc} = 0$. Thus we are left with one equation of motion in two accelerations, which is often rewritten slightly as

$$F = m\lambda_m \left(a_{cc} + \lambda a_{ss}\right) \tag{2.28}$$

Substituting the definitions of a_{cc}, a_{ss}, D_c, and D_s and letting $\lambda_m = 1$ by convention, we get a full equation of motion:

$$F = m\left(\frac{\partial v}{\partial t} + v \cdot (\nabla v) + \lambda\left(u \cdot (\nabla u) + D\nabla^2 u\right)\right) \tag{2.29}$$

which is called the **Nelson-Newton equation.**

2.3 Mechanics 3: The Fokker-Planck equations

The Nelson-Newton equation is one of the two fundamental stochastic equations from which the Schrödinger equation can be derived. The other is similar to a continuity equation, and deals not only with functions of the stochastic variable x and its velocities, but also the *probability distribution* for these functions. The evolution of this probability distribution is goverened by the **Fokker-Planck equations**. We will quickly derive these here, as this derivation will be reexamined later in curved space.

Consider a deterministic function f of a stochastic variable x. Since f depends on a stochastic variable, its differential change $df(x)$ depends on its higher order derivatives:

$$df(x) = \nabla f \cdot dx + \frac{1}{2}\nabla^2 f dx^2 + ... \tag{2.30}$$

Substituting the stochastic differential equation for x reveals not one, but two terms proportional to dt:

$$\begin{align} df(x) &= \nabla f \cdot \left((v+u)dt + \sqrt{2D}dW\right) + \frac{1}{2}\nabla^2 f \left((v+u)dt + \sqrt{2D}dW\right)^2 + .. \tag{2.31} \\ &= \nabla f \cdot \sqrt{2D}dW + \nabla f \cdot (v+u)dt + D\nabla^2 f dt + .. \tag{2.32} \end{align}$$

We want to get at the probability distribution for f. The most natural way to do this is to compute the *ensemble average* of f. This trick also gets rid of the troublesome dW term; since the dW kicks are normally distributed, their contribution will disappear when averaged. Thus calculating the expectation value of df gives us an equation proportional to dt:

$$\langle df \rangle = d\langle f \rangle = \int p(x,t) df(x) dx \tag{2.33}$$

$$= \int p(x,t)\left(\nabla f \cdot (v+u) + D\nabla^2 f\right) dtdx \tag{2.34}$$

Instruting the mathematicians to avert their gaze and dividing through by dt, we get an equation for $\frac{d\langle f \rangle}{dt}$:

$$\frac{d\langle f\rangle}{dt} = \int p(x,t)\left(\nabla f \cdot (v+u) + D\nabla^2 f\right) dx \tag{2.35}$$

However, this is an equation involving the derivatives of f, and is therefore does not help us determine the dynamics of p. To fix this, we integrate both components of the integral by parts, using the fact that $\lim_{x\to\pm\infty} p(x,t) = 0$:

$$\frac{d\langle f\rangle}{dt} = \int \left(-\nabla \cdot p(x,t)(v+u) + D\nabla^2 p(x,t)\right) f dx \tag{2.36}$$

But we could also calculate $\frac{d\langle f\rangle}{dt}$ by simply taking the time derivative of the earlier expression:

$$\begin{aligned}\frac{d\langle f\rangle}{dt} &= \frac{d}{dt}\int p(x,t) df(x) dx &(2.37)\\ &= \int \frac{\partial p}{\partial t} df(x) dx &(2.38)\end{aligned}$$

Settings these integrands equal and rearranging gives us the **forward Fokker-Planck equation**:

$$\frac{\partial p}{\partial t} + \nabla \cdot p(x,t)(v+u) - D\nabla^2 p(x,t) = 0 \tag{2.39}$$

We will not derive the **backward Fokker-Planck equation** here, as the steps are very similar. Instead, we will simply state it:

$$\frac{\partial p}{\partial t} + \nabla \cdot p(x,t)(v-u) + D\nabla^2 p(x,t) = 0 \tag{2.40}$$

The astute reader may notice that these equations are seperable in v and u. Adding them gives us the **second equation of motion** that we mentioned earlier, which looks like a continuity equation:

$$\frac{\partial p}{\partial t} + \nabla \cdot (p(x,t)v) = 0 \tag{2.41}$$

Subtracting them does not give us an equation of motion, but rather a useful expression for u in terms of p:

$$\nabla \cdot (p(x,t)u) - D\nabla^2 p = 0 \tag{2.42}$$

This can be integrated, assuming u has no rotational component [8], to give

$$u = D\frac{\nabla p}{p} \tag{2.43}$$

which is known as Fick's law of diffusion.

As a closing note, it is often useful to talk about the **probability current** of a distribution $p(x, t)$ that follows a Fokker-Planck equation. The definition of a probability current J for a distribution p usually follows from assuming that probability is conserved, which gives the continuity equation

$$\frac{\partial p}{\partial t} + \nabla \cdot J = 0 \tag{2.44}$$

Note that the forward Fokker-Planck equation takes this exact form if we define the probability current as

$$J(x, t) = p(x, t)(v + u) - D\nabla p(x, t) \tag{2.45}$$

We will call this the **Fokker-Planck probability current**.

2.4 From Nelson to Schrödinger

We have derived the following equations of motion:

$$F = m\left(\frac{\partial v}{\partial t} + v \cdot (\nabla v) + \lambda\left(u \cdot (\nabla u) + D\nabla^2 u\right)\right) \tag{2.46}$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (p(x, t)v) = 0 \tag{2.47}$$

Turnings these into the Schrödinger equation is a two-step process. First, we will rewrite v and u in terms of p and s, which will become the magnitude and phase of our wavefunction, and integrate the first equation; once this is done, we can separate the equations and derive the Schrödinger equation via a change of variables.

Our first goal is to integrate the Nelson-Newton equation. This is already nearly done; in fact, if we consider a conservative force $F = -\nabla V$, then all but one terms can be written as a gradient already:

$$-\nabla V = m\left(\frac{\partial v}{\partial t} + \nabla\left(\frac{v\cdot v}{2}\right) + \lambda\left(\nabla\left(\frac{u\cdot u}{2}\right) + \nabla(D\nabla\cdot u)\right)\right) \tag{2.48}$$

Writing the final term as a gradient is somewhat difficult, and is not strictly required to re-derive the Schroödinger equation. However, since we will be considering only spherically symmetric potentials, we can safely assume in our case that v has no rotational component. Thus we can write v as the gradient of some function S. It turns out that the most convenient way to do this is as follows:

$$v = 2D\nabla S \tag{2.49}$$

Writing v in this way lets us finally integrate the Nelson-Newton equation:

$$-V = m\left(2D\frac{\partial S}{\partial t} + 4D^2\frac{\nabla S\cdot\nabla S}{2} + \lambda\left(\frac{u\cdot u}{2} + D\nabla\cdot u\right)\right) \tag{2.50}$$

For good measure, we should make the same substitution in the continuity equation:

$$\frac{\partial p}{\partial t} + 2D\nabla\cdot(p(x,t)\nabla S) = 0 \tag{2.51}$$

Now we just need to substitute our expression for u in terms of p into the first equation, and we will have our differential equations entirely in terms of p and S. It turns out there is a clever way to do this that combines the two u-dependent terms into one. We start with direct substitution:

$$\frac{u\cdot u}{2} + D\nabla\cdot u = D^2\frac{\nabla p\cdot\nabla p}{2p^2} + D^2\nabla\cdot\frac{\nabla p}{p} \tag{2.52}$$

$$= D^2\left(\frac{\nabla p\cdot\nabla p}{2p^2} + \frac{\nabla^2 p}{p} - \frac{\nabla p\cdot\nabla p}{p^2}\right) \tag{2.53}$$

$$= D^2\left(\frac{\nabla^2 p}{p} - \frac{\nabla p\cdot\nabla p}{2p^2}\right) \tag{2.54}$$

The astute reader may smell a gradient lurking. And in fact, this is *almost* equal to

$$\nabla^2\sqrt{p} = \nabla\cdot\left(\frac{1}{2}\frac{\nabla p}{\sqrt{p}}\right) \tag{2.55}$$

$$= \frac{1}{2}\left(\frac{\nabla^2 p}{\sqrt{p}} - \frac{1}{2}\frac{(\nabla p)^2}{p^{3/2}}\right) \tag{2.56}$$

This is almost the function we want, but it's off by a factor of $\frac{2}{\sqrt{p}}$. We will simply multiply this factor in:

$$2\frac{\nabla^2\sqrt{p}}{\sqrt{p}} = \frac{\nabla^2 p}{p} - \frac{1}{2}\frac{(\nabla p)^2}{p^2} \tag{2.57}$$

In this way we can rewrite the terms in the equation of motion that depend on u as one term:

$$\frac{u^2}{2} + D\nabla \cdot u = 2D^2\frac{\nabla^2\sqrt{p}}{\sqrt{p}} \tag{2.58}$$

Substituting this back in and rearranging slightly, we have our equations of motion in terms of only p and S:

$$m\left(2D\frac{\partial S}{\partial t} + 4D^2\frac{\nabla S \cdot \nabla S}{2} + 2D^2\lambda\frac{\nabla^2\sqrt{p}}{\sqrt{p}}\right) + V = 0 \tag{2.59}$$

$$\frac{\partial p}{\partial t} + 2D\nabla \cdot (p\nabla S) = 0 \tag{2.60}$$

We can now separate these equations by changing variables to the wavefunctions ψ_+ and ψ_-:

$$\psi_+ = \sqrt{p}e^{S/\sqrt{\lambda}} \tag{2.61}$$

$$\psi_- = \sqrt{p}e^{-S/\sqrt{\lambda}} \tag{2.62}$$

which gives the substitutions

$$p = \psi_+\psi_- \tag{2.63}$$

$$S = \frac{\sqrt{\lambda}}{2}\ln\left(\frac{\psi_+}{\psi_-}\right) \tag{2.64}$$

Let us start our algebra with the simpler continuity equation:

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + 2D\nabla \cdot \left(\psi_+\psi_-\nabla\frac{\sqrt{\lambda}}{2}\ln\left(\frac{\psi_+}{\psi_-}\right)\right) = 0 \tag{2.65}$$

Pulling out constants:

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\nabla \cdot \left(\psi_+\psi_-\nabla\ln\left(\frac{\psi_+}{\psi_-}\right)\right) = 0 \tag{2.66}$$

Evaluating the inner gradient:

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\nabla \cdot \left(\psi_+\psi_-\frac{\psi_-}{\psi_+}\nabla\left(\frac{\psi_+}{\psi_-}\right)\right) = 0 \tag{2.67}$$

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\nabla \cdot \left(\psi_-^2\nabla\left(\frac{\psi_+}{\psi_-}\right)\right) = 0 \tag{2.68}$$

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\nabla \cdot \left(\psi_-^2\left(\frac{\nabla\psi_+}{\psi_-} - \frac{\psi_+\nabla\psi_-}{\psi_-^2}\right)\right) = 0 \tag{2.69}$$

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\nabla \cdot (\psi_-\nabla\psi_+ - \psi_+\nabla\psi_-) = 0 \tag{2.70}$$

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\left(\nabla\psi_- \cdot \nabla\psi_+ + \psi_-\nabla^2\psi_+ - \nabla\psi_+ \cdot \nabla\psi_- - \psi_+\nabla^2\psi_-\right) = 0 \tag{2.71}$$

$$\frac{\partial(\psi_+\psi_-)}{\partial t} + \sqrt{\lambda}D\left(\psi_-\nabla^2\psi_+ - \psi_+\nabla^2\psi_-\right) = 0 \tag{2.72}$$

Expanding the time derivative:

$$\frac{\partial\psi_+}{\partial t}\psi_- + \frac{\partial\psi_-}{\partial t}\psi_+ + \sqrt{\lambda}D\left(\psi_-\nabla^2\psi_+ - \psi_+\nabla^2\psi_-\right) = 0 \tag{2.73}$$

This is as far as we can expand the continuity equation. The second equation is more complicated, so let's look at each term by itself:

$$2Dm\frac{\partial S}{\partial t} = 2Dm\frac{\partial}{\partial t}\frac{\sqrt{\lambda}}{2}\ln\left(\frac{\psi_+}{\psi_-}\right) \tag{2.74}$$

Pulling out a constant:

$$2Dm\frac{\partial S}{\partial t} = \sqrt{\lambda}Dm\frac{\partial}{\partial t}\ln\left(\frac{\psi_+}{\psi_-}\right) \tag{2.75}$$

Doing the time derivative:

$$2Dm\frac{\partial S}{\partial t} = \sqrt{\lambda}Dm\frac{\psi_-}{\psi_+}\frac{\partial}{\partial t}\left(\frac{\psi_+}{\psi_-}\right) \tag{2.76}$$

$$= \sqrt{\lambda}Dm\frac{\psi_-}{\psi_+}\left(\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_-} - \frac{\partial\psi_-}{\partial t}\frac{\psi_+}{\psi_-^2}\right) \tag{2.77}$$

$$= \sqrt{\lambda}Dm\left(\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+} - \frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-}\right) \tag{2.78}$$

That's all we can do for the first term. Now for the second term:

$$2D^2m(\nabla S)^2 = 2D^2m\left(\nabla\frac{\sqrt{\lambda}}{2}\ln\left(\frac{\psi_+}{\psi_-}\right)\right)^2 \tag{2.79}$$

Pulling out the constant:

$$2D^2m(\nabla S)^2 = \frac{\lambda D^2m}{2}\left(\nabla\ln\left(\frac{\psi_+}{\psi_-}\right)\right)^2 \tag{2.80}$$

Doing the gradient:

$$2D^2m(\nabla S)^2 = \frac{\lambda D^2m}{2}\left(\frac{\psi_-}{\psi_+}\nabla\left(\frac{\psi_+}{\psi_-}\right)\right)^2 \tag{2.81}$$

$$= \frac{\lambda D^2m}{2}\left(\frac{\psi_-}{\psi_+}\left(\frac{\nabla\psi_+}{\psi_-} - \frac{\psi_+\nabla\psi_-}{\psi_-^2}\right)\right)^2 \tag{2.82}$$

$$= \frac{\lambda D^2m}{2}\left(\frac{\nabla\psi_+}{\psi_+} - \frac{\nabla\psi_-}{\psi_-}\right)^2 \tag{2.83}$$

$$= \frac{\lambda D^2m}{2}\left(\frac{(\nabla\psi_+)^2}{\psi_+^2} - 2\frac{\nabla\psi_+\cdot\nabla\psi_-}{\psi_+\psi_-} + \frac{(\nabla\psi_-)^2}{\psi_-^2}\right) \tag{2.84}$$

Now the last term:

$$2D^2 m\lambda \frac{\nabla^2 \sqrt{p}}{\sqrt{p}} = 2D^2 m\lambda \frac{\nabla^2 \sqrt{\psi_+\psi_-}}{\sqrt{\psi_+\psi_-}} \tag{2.85}$$

Nothing to do here but dive into the Laplacian:

$$2D^2 m\lambda \frac{\nabla^2 \sqrt{p}}{\sqrt{p}} = D^2 m\lambda \frac{\nabla^2 \left(\psi_+\psi_-\right)}{\psi_+\psi_-} \tag{2.86}$$

$$= D^2 m\lambda \frac{(\nabla^2\psi_+)\psi_- + 2\nabla\psi_+ \cdot \nabla\psi_- + \psi_+(\nabla^2\psi_-)}{\psi_+\psi_-} \tag{2.87}$$

$$= D^2 m\lambda \left(\frac{\nabla^2\psi_+}{\psi_+} + 2\frac{\nabla\psi_+ \cdot \nabla\psi_-}{\psi_+\psi_-} + \frac{\nabla^2\psi_-}{\psi_-}\right) \tag{2.88}$$

Throwing this Laplacian into Mathematica:

$$\nabla^2 \sqrt{\psi_+\psi_-} = \frac{1}{4(\psi_+\psi_-)^{3/2}} (2\psi_+\psi_- \left(\psi_+\nabla^2\psi_- + \nabla\psi_+ \cdot \nabla\psi_-\right) - \psi_-^2 \left(-2\psi_+\nabla^2\psi_+ + (\nabla\psi_+)^2\right) - \psi_+^2(\nabla\psi_-)^2$$

$$= \frac{1}{4(\psi_+\psi_-)^{3/2}} \left(2\psi_+\psi_-\nabla\psi_+ \cdot \nabla\psi_- + 2\psi_+^2\psi_-\nabla^2\psi_- + 2\psi_-^2\psi_+\nabla^2\psi_+ - \psi_-^2(\nabla\psi_+)^2 - \psi_+^2(\nabla\psi_-)^2\right.$$

Dividing through by $\sqrt{\psi_+\psi_-}$ and cancelling some terms gives us a nicer looking equation:

$$\frac{\nabla^2 \sqrt{\psi_+\psi_-}}{\sqrt{\psi_+\psi_-}} = \frac{1}{4(\psi_+\psi_-)^2} (2\psi_+\psi_-\nabla\psi_+ \cdot \nabla\psi_- \tag{2.89}$$

$$+2\psi_+^2\psi_-\nabla^2\psi_- + 2\psi_-^2\psi_+\nabla^2\psi_+ - \psi_-^2(\nabla\psi_+)^2 - \psi_+^2(\nabla\psi_-)^2) \tag{2.90}$$

$$= \left(\frac{\nabla\psi_+ \cdot \nabla\psi_-}{2\psi_+\psi_-} + \frac{\nabla^2\psi_-}{2\psi_-} + \frac{\nabla^2\psi_+}{2\psi_+} - \frac{(\nabla\psi_+)^2}{4\psi_+} - \frac{(\nabla\psi_-)^2}{4\psi_-^2}\right) \tag{2.91}$$

Plugging this back into the term:

$$2D^2 m\lambda \frac{\nabla^2 \sqrt{p}}{\sqrt{p}} = D^2 m\lambda \left(\frac{\nabla\psi_+ \cdot \nabla\psi_-}{\psi_+\psi_-} + \frac{\nabla^2\psi_-}{\psi_-} + \frac{\nabla^2\psi_+}{\psi_+} - \frac{(\nabla\psi_+)^2}{2\psi_+} - \frac{(\nabla\psi_-)^2}{2\psi_-^2}\right) \tag{2.92}$$

Finally, substituting all these terms back into the original equation gives us the formidable formula

$$\sqrt{\lambda}Dm\left(\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+}-\frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-}\right)+\frac{\lambda D^2 m}{2}\left(\frac{(\nabla\psi_+)^2}{\psi_+^2}-2\frac{\nabla\psi_+\cdot\nabla\psi_-}{\psi_+\psi_-}+\frac{(\nabla\psi_-)^2}{\psi_-^2}\right) \tag{2.93}$$

$$+D^2 m\lambda\left(\frac{\nabla\psi_+\cdot\nabla\psi_-}{\psi_+\psi_-}+\frac{\nabla^2\psi_-}{\psi_-}+\frac{\nabla^2\psi_+}{\psi_+}-\frac{(\nabla\psi_+)^2}{2\psi_+}-\frac{(\nabla\psi_-)^2}{2\psi_-^2}\right)=0 \tag{2.94}$$

Thankfully, the entire middle term is cancelled by parts of the last term. After cancelling, the equation now fits on the page!

$$\sqrt{\lambda}Dm\left(\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+}-\frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-}\right)+D^2 m\lambda\left(\frac{\nabla^2\psi_-}{\psi_-}+\frac{\nabla^2\psi_+}{\psi_+}\right)=0 \tag{2.95}$$

We will separate this together with the reframed continuity equation:

$$\frac{\partial\psi_+}{\partial t}\psi_-+\frac{\partial\psi_-}{\partial t}\psi_++\sqrt{\lambda}D\left(\psi_-\nabla^2\psi_+-\psi_+\nabla^2\psi_-\right)=0 \tag{2.96}$$

First, notice the abundance of terms similar to $\frac{\partial\psi_\pm}{\psi_\pm}$ in the first equation and the similar abundance of $(\partial\psi_\pm)\psi_\mp$ terms in the second equation. We can convert one to another by multiplying or dividing through by $\psi_+\psi_-$. In this spirit of separation of variables, let's choose to divide the continuity equation by $\psi_+\psi_-$:

$$\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+}+\frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-}+\sqrt{\lambda}D\left(\frac{\nabla^2\psi_+}{\psi_+}-\frac{\nabla^2\psi_-}{\psi_-}\right)=0 \tag{2.97}$$

Multiplying by $\sqrt{\lambda}Dm$ gives us something similar to the first equation, but antisymmetric in ψ_+ and ψ_-:

$$\sqrt{\lambda}Dm\left(\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+}+\frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-}\right)+\lambda D^2 m\left(\frac{\nabla^2\psi_+}{\psi_+}-\frac{\nabla^2\psi_-}{\psi_-}\right)=0 \tag{2.98}$$

Adding and subtracting this from the first equation eliminates the terms that depend on ψ_- and ψ_+ respectively:

$$2\sqrt{\lambda}Dm\frac{\partial\psi_+}{\partial t}\frac{1}{\psi_+} + 2D^2m\lambda\frac{\nabla^2\psi_+}{\psi_+} + V = 0 \tag{2.99}$$

$$-2\sqrt{\lambda}Dm\frac{\partial\psi_-}{\partial t}\frac{1}{\psi_-} + 2D^2m\lambda\frac{\nabla^2\psi_-}{\psi_-} + V = 0 \tag{2.100}$$

We now see the light. Moving the time derivatives to the other side and multiplying through by ψ_+ and ψ_- respectively, we get something very reminiscent of the Schrödinger equation:

$$-2\sqrt{\lambda}Dm\frac{\partial\psi_+}{\partial t} = 2D^2m\lambda\nabla^2\psi_+ + V\psi_+ \tag{2.101}$$

$$2\sqrt{\lambda}Dm\frac{\partial\psi_-}{\partial t} = 2D^2m\lambda\nabla^2\psi_- + V\psi_- \tag{2.102}$$

The classical Schrödinger equation follows if we choose $\lambda = -1$ and $D = \frac{\hbar}{2m}$:

$$-i\hbar\frac{\partial\psi_+}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_+ + V\psi_+ \tag{2.103}$$

$$i\hbar\frac{\partial\psi_-}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_- + V\psi_- \tag{2.104}$$

2.5 The Stochastic Harmonic Oscillator

The harmonic oscillator is one of the simplest systems in classical quantum mechanics. Similarly, the stochastic harmonic oscillator is described by the Ornstein-Uhlenbeck process, defined by the following stochastic differential equation:

$$dx = -axdt + \sqrt{2D}dW \tag{2.105}$$

From our derivation of the Schrödinger equation, we know that we should let $D = \frac{\hbar}{2m}$. We will also find that the velocity a corresponds to the frequency of the harmonic oscillator. Knowing all this, we define the stochastic harmonic oscillator as the system

$$dx = -\omega x dt + \sqrt{\frac{\hbar}{m}} dW \tag{2.106}$$

The connection between this process and the quantum harmonic oscillator comes from considering an ensemble of processes that obey this differential equation. If this ensemble has a delta function distribution $p(x, t) = \delta(x_0)$ at time t_0, then at time t it takes the form [9]

$$p(x, t) = \sqrt{\frac{\omega m}{2\pi\hbar\left(1 - e^{-2a(t-t_0)}\right)}} e^{-\frac{\omega m}{2\pi\hbar\left(1-e^{-2\omega(t-t_0)}\right)}\left(x - x_0 e^{-\omega(t-t_0)}\right)^2} \tag{2.107}$$

This is a Gaussian with mean $x_0 e^{-\omega(t-t_0)}$ and width $\frac{\omega}{D(1-e^{-2\omega(t-t_0)})}$. Note that as $t \to \infty$, this becomes nothing but the ground state of the quantum harmonic oscillator. Furthermore, the speed of this convergence increases with ω. This is an important result: *low frequency oscillators converge to equilibrium slower than high frequency oscillators.*

The rate at which a stochastic harmonic oscillator converges to equilibrium is crucial in studying the quadrupole discrepancy. Thus it is worth studying in more depth. We will present basic theoretical and numerical results on this topic.

2.5.1 Theoretical Results

First Crossing Time Distribution

A common measure of how fast a stochastic process reaches a certain point q is the **first crossing time distribution** of the probability distribution $p(x, t)$ across that point. The first crossing time across q for a single stochastic process x starting at x_0 is the first time that that process reaches q. More generally, for a probability distribution $p(x, t)$ of some stochastic process, we can define the *first crossing density* $p_{fc}(X, t)$ as the probability density that a stochastic process first crosses X at time t. Assuming the initial distribution $p(x_0, t_0)$ starts entirely on one side of X, this is just the magnitude of the Fokker-Planck probability current through X at time t:

$$p_{fc}(X, t) = |J(X, t)| = |p(X, t)(v + u) - D\nabla p(X, t)| \tag{2.108}$$

To calculate the first crossing density for the stochastic harmonic oscillator, we substitute $v + u = -\omega x$ and $D = \frac{\hbar}{2m}$

$$p_{fc}(X,t) = \left| -\omega x p(X,t) - \frac{\hbar}{2m}\nabla p(x,t) \right| \tag{2.109}$$

When a stochastic system evolves to some equilibrium, the first crossing time of the mean of that equilibrium is often interesting. For the stochastic harmonic oscillator, this means we want to calculate the first crossing density at $X = 0$. Substituting this simplifies the first crossing density greatly:

$$p_{fc}(X,t) = \frac{\hbar}{2m}\nabla p(x,t) \tag{2.110}$$

We can calculate this density analytically for a stochastic harmonic oscillator that starts with a distribution $p(x,0) = \delta(x_0)$:

$$p_{fc}(0,t) = \frac{\hbar}{2m}\frac{\partial p}{\partial x}(0,t) = \frac{\hbar}{m}\sqrt{\frac{\omega m}{\pi\hbar\left(1-e^{-2\omega t}\right)}}\frac{2\omega m x_0 e^{-\omega t}}{\hbar\left(1-e^{-2\omega t}\right)}e^{-\frac{\omega m}{\hbar\left(1-e^{-2\omega t}\right)}x_0^2 e^{-2\omega t}} \tag{2.111}$$

Future research might attempt to calculate the peak of this distribution as a function of ω by taking its derivative, or find its average by calculating its integral. Basic attempts at differentiation via SymPy 1.9 and integration via the modified Risch-Normal algorithm did not converge in a reasonable time.

Bhattacharyya distance

Another way to calculate how fast a stochastic harmonic oscillator converges to its quantum counterpart is to compute a similarity measure between the probability distributions. One measure that can be calculated analytically is the Bhattacharyya distance. The Bhattacharyya distance between two distributions $f(x,t)$ and $g(x,t)$ is defined as

$$D_B(f,g) = \int_{-\infty}^{\infty} \sqrt{f(t)g(t)}dt \tag{2.112}$$

Let us consider the probability distributions $|\Psi(x)|^2$ and $p(x,t)$ for the ground states of the classical and stochastic harmonic oscillators. The classical distribution $|\Psi(x)|^2$ is a Gaussian with mean $\mu_C = 0$ and standard deviation $\sigma_C = \sqrt{\frac{\hbar}{2m\omega}}$, while the stochastic distribution is a Gaussian with mean $\mu_S = x_0 e^{-\omega t}$ and standard deviation $\sigma_C = \sqrt{\frac{\hbar(1-e^{-2\omega t})}{2m\omega}}$. Thus their product $|\Psi(x)|^2 p(x,t)$ is also a Gaussian with mean

$$\mu_P = \frac{\mu_C\sigma_S^2 + \mu_S\sigma_C^2}{\sigma_C^2+\sigma_S^2} = \frac{x_0 e^{-\omega t}\frac{\hbar}{2m\omega}}{\frac{\hbar}{2m\omega}\left(2-e^{-2\omega t}\right)} = \frac{x_0 e^{-\omega t}}{2-e^{-2\omega t}} \tag{2.113}$$

and standard deviation

$$\sigma_P = \sqrt{\frac{(\sigma_S\sigma_C)^2}{\sigma_S^2 + \sigma_C^2}} = \sqrt{\frac{\frac{\hbar^2(1-e^{-2\omega t})}{4m^2\omega^2}}{\frac{\hbar}{2m\omega}\left(2 - e^{-2\omega t}\right)}} = \sqrt{\frac{\hbar}{2m\omega}\frac{1 - e^{-2\omega t}}{2 - e^{-2\omega t}}} \tag{2.114}$$

Thus the Bhattacharyya distance is

$$D_B\left(|\Psi(x)|^2, p(x,t)\right) = \int_{-\infty}^{\infty}\left(\frac{m\omega}{\hbar\pi}\sqrt{\frac{1}{1-e^{-2\omega t}}}e^{-\frac{1}{2}\left(\frac{x-\mu_P}{\sigma_P}\right)^2}\right)^{1/2} dx \tag{2.115}$$

$$= \sqrt{\frac{m\omega}{\hbar\pi}}(1-e^{-2\omega t})^{-1/4}\int_{-\infty}^{\infty} e^{-\frac{1}{4}\left(\frac{x-\mu_P}{\sigma_P}\right)^2} dx \tag{2.116}$$

Substituting the mean and standard deviation gives a refreshingly simple form:

$$D_B\left(|\Psi(x)|^2, p(x,t)\right) = \sqrt{\frac{4m\omega}{\hbar}}(1-e^{-2\omega t})^{-1/4}\sqrt{\frac{\hbar}{2m\omega}\frac{1-e^{-2\omega t}}{2-e^{-2\omega t}}} = \sqrt{\frac{2\sqrt{1-e^{-2\omega t}}}{2-e^{-2\omega t}}} \tag{2.117}$$

As expected, this correlation is 0 at $t = 0$ and goes to 1 as $t \to \infty$.

2.5.2 Numerical Results

The probability distribution $p(x,t)$ of a stochastic process can be approximated by simulating a large number of particles evolving with the stochastic differential equation. Simulating an ensemble of 10^6 particles in natural units with $m = 1$ at several different ωs illustrates how convergence is faster for higher ω. The full animation cannot be embedded in this paper, but slices taken at specific times can, and this is shown in Figure 2.1.

The average first crossing time of $x = 00$ for similar ensembles at different ωs was also simulated. The relationship between ω and observed first crossing time fits best to $\frac{1}{\omega}$ with $r^2 = 1.7 \cdot 10^{-3}$, as seen in in Figure 2.2.

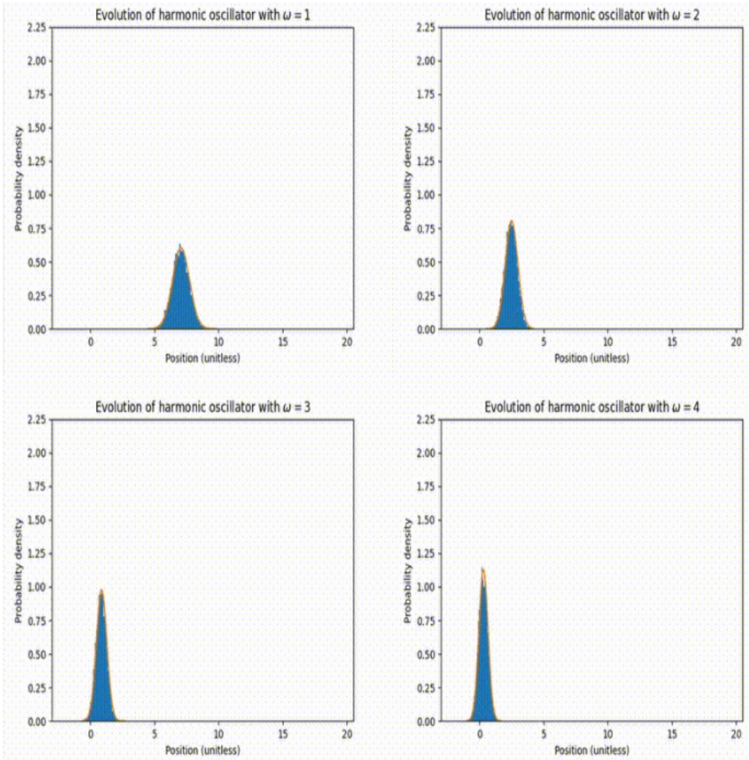

Figure 2.1: A snapshot of ensembles evolving as stochastic harmonic oscillators at time $t = 0.5$ from an original position of $x_0 = 15$. Simulated ensemble is blue; theoretical prediction is orange.

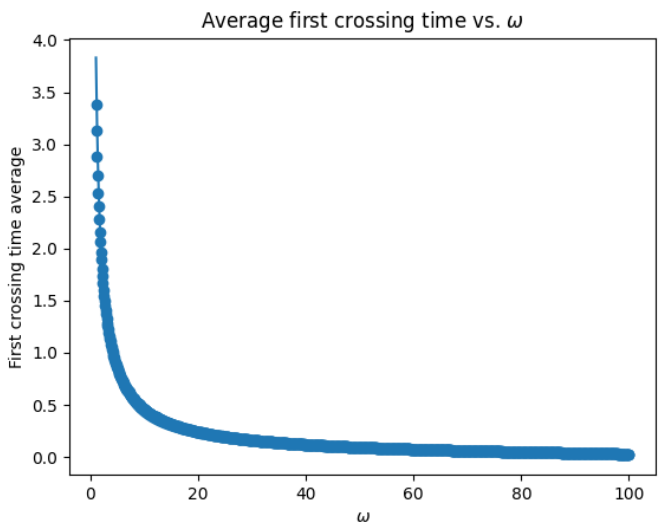

Figure 2.2: Average crossing time of 0 for stochastic harmonic oscillators starting at $x_0 = 15$ as a function of ω.

3 Inflationary Cosmology

3.1 Schrödinger equation in expanding spacetime

In this paper, we consider the flat metric

$$ds^2 = dt^2 - a(t)^2 dx^2 \tag{3.1}$$

and the natural free-field Lagrangian density for this metric (where we have included the integration factor $\sqrt{|g|}$ inside the density):

$$\mathcal{L} = \frac{\sqrt{|g|}}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta \phi \tag{3.2}$$

For simplicity, we will generally choose a frame and expand this Lagrangian density into temporal and spatial components:

$$\begin{aligned} L &= \frac{a(t)^3}{2} \left(\dot{\phi}^2 - \frac{1}{a(t)^2} (\nabla \phi)^2 \right) && (3.3) \\ &= \frac{1}{2} \left(a(t)^3 \dot{\phi}^2 - a(t) (\nabla \phi)^2 \right) && (3.4) \end{aligned}$$

We can do two things with this Lagrangian. First, we can solve the Euler-Lagrange equations to get the wave equation for ϕ:

$$
\begin{aligned}
\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} &= \frac{\partial L}{\partial \phi} \\
\implies \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}} + \nabla \cdot \frac{\partial L}{\partial \nabla \phi} &= 0 \\
\implies \frac{\partial}{\partial t}\left(a(t)^3 \dot{\phi}\right) - \nabla \cdot a(t) \nabla \phi &= 0 \\
\implies a(t)^3 \ddot{\phi} + 3a(t)^2 \dot{a}(t) \dot{\phi} - a(t) \nabla^2 \phi &= 0
\end{aligned}
$$

It is convenient to divide this equation by $a(t)^3$, since this turns the clunky factor of $\dot{a}(t)$ into the Hubble paramter $H := \frac{\dot{a}(t)}{a(t)}$:

$$
\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a(t)^2}\nabla^2 \phi = 0 \tag{3.5}
$$

Second, having this Lagrangian lets us derive the Hamiltonian for ϕ via the conjugate momentum $\mu = \frac{\partial L}{\partial \dot{\phi}} = a(t)^3 \dot{\phi}$:

$$
H = a(t)^3 \dot{\phi}^2 - L = \frac{1}{2}\left(a(t)^3 \dot{\phi}^2 + a(t)(\nabla \phi)^2\right) \tag{3.6}
$$

$$
= \frac{\mu^2}{2a(t)^3} + a(t)(\nabla \phi)^2 \tag{3.7}
$$

We can turn this into the functional Schrödinger equation for a functional $\psi(\phi(x, t), t)$ via the usual quantization of $\hat{\mu} = -i\frac{\delta}{\delta \phi}$, $\hat{\phi} = \phi$:

$$
i\frac{\partial \psi}{\partial t} = -\frac{1}{2a(t)^3}\frac{\delta^2 \psi}{\delta \phi^2} + a(t)(\nabla \phi)^2 \psi \tag{3.8}
$$

In order to apply our theory of stochastic harmonic oscillators to the functional ψ, we need to split it into its application on their Fourier modes ϕ_k:

$$
\phi_k(k) = \frac{1}{(2\pi)^{3/2}} \int \phi(x) e^{-ix\cdot k} d^3x \tag{3.9}
$$

Each mode then satisfies its own Schrödinger equation. However, since the gradient of any mode ϕ_k is just $\nabla \phi_k = k\phi_k$, the Schrödinger equation is greatly simplified:

$$i\frac{\partial\psi[\phi_k]}{\partial t} = -\frac{1}{2a(t)^3}\frac{\partial^2\psi[\phi_k]}{\partial\phi_k^2} + a(t)k^2\phi_k^2\psi[\phi_k] \tag{3.10}$$

This looks very similar to the Schrödinger equation for a harmonic oscillator. We will exploit this similarity when we describe the same modes in terms of stochastic mechanics.

3.2 Anisitropies in the Cosmic Microwave Background

Our goal is to describe the temperature anisotropy in the cosmic microwave background (CMB) in terms of stochastic harmonic oscillators. We have established the theory of such oscillators and discovered how to write the modes of the inflaton field, but we have not yet discussed how the fluctuations we hope to explain are described. Here, we will give a quick overview of the cosmology behind temperature fluctuations in the CMB. This discussion will roughly follow that of Valentini [4].

First, we will define temperature anisotropy. Every measurement of the CMB measures a function $T(\theta, \phi)$ of the temperature at some solid angle. Denoting the average temperature across the whole sky as $\overline{T}$, we define the **temperature anisotropy** ΔT as

$$\Delta T(\theta, \phi) = \frac{T(\theta, \phi) - \overline{T}}{\overline{T}} \tag{3.11}$$

This anisotropy is often written in terms of spherical harmonics with coefficients $c_{\ell m}$:

$$\Delta T(\theta, \phi) = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta, \phi) \tag{3.12}$$

Any measurement of the temperature perturbation $\Delta T(\theta, \phi)$ thus consists of one set of coefficients $c_{\ell m}$. It is typically assumed that for every ℓ, the $c_{\ell m}$ are drawn from the isotropic probability distribution. The **angular power spectrum** of this probability distribution is written as

$$C_\ell := \langle |c_{\ell m}^2| \rangle \tag{3.13}$$

where $\langle\rangle$ is an ensemble average.

In the same way, we can break down the Ricci curvature R in terms of its Fourier components R'_k. However, it is more useful to work with them in terms of **classical curvature coefficients** R_k, defined as

$$R_k = \frac{1}{4}\left(\frac{a}{k}\right)^2 R'_k \tag{3.14}$$

These components are also presumed to come from a probability distribution. We define the **power spectrum of the curvature perturbations** P_k in terms of the ensemble average of the squared magnitudes of the R_k and the normalization volume V:

$$P_k = \frac{4\pi k^3}{V}\left\langle |R_k|^2 \right\rangle \tag{3.15}$$

For simplicity, we will assume that gravitational waves have a negligible effect on the curvature. Then there is a simple relation between the angular power spectrum and the curvature power spectrum in terms of the transfer function $T(k,\ell)$:

$$C_\ell = \frac{1}{2\pi^2}\int \frac{T(k,\ell)^2}{k} P_k dk \tag{3.16}$$

This relation between the temperature anisotropy and the power spectrum of curvature perturbations is key. If we can express the power spectrum of curvature perturbations in terms of the widths of modes of the inflaton field, we can make observable conclusions about how stochastic evolution of the modes to equilibrium affects the temperature anisotropy.

We can do this by analyzing the inflaton field in a perturbative way. Since the universe on the largest scales is mostly homogeneous, the inflaton field that generated this homogeneity should be mostly homogeneous as well. Thus we generally expand the inflaton field ϕ into a homogeneous part ϕ_0 and a small position-dependent perturbation ϕ^p:

$$\phi(x,t) = \phi_0(t) + \phi^p(x,t) \tag{3.17}$$

The perturbation field $\phi^p(x,t)$ creates the curvature perturbations we measure today. As such, one should expect the classical curvature coefficient R_k to be related to the kth Fourier mode of $\phi^p(x,t)$. Such a relation exists, but requires a slightly deeper discussion of inflation.

Inflation is typically modelled as a short-lived exponential growth in the scale factor $a(t)$. That is, during inflation, we have $a(t) \sim e^{Ht}$, where $H = \frac{\dot{a}}{a}$. Furthermore, inflation occurs so quickly that H is approximately constant for the duration.

Consider a Fourier mode $\phi^p_k(x,t)$ of the perturbation field in this environment. On a comoving surface, this mode has wavelength $\frac{2\pi a(t)}{k}$. When this wavelength exceeds the radius of causality H^{-1}, distant parts of the mode are no longer able to communicate with each other; thus the perturbation caused by this mode quickly "freezes". The "frozen" mode is then stretched out by inflation, distributing it homogeneously across space. Finally, after inflation has stopped, the wavelength finally becomes smaller than H^{-1} and the perturbations can again propagate, causing anisotropies in the CMB.

The time at which the wavelength of the kth mode exceeds the radius of causality is called the **exit time** $t_{\text{exit}}(k)$. Thus we should expect the curvature perturbation R_k caused by $\phi_k^p(x,t)$ to be related to $\phi_k^p(x,t)$ evaluated at a time slightly after $t_{\text{exit}}(k)$, when the perturbation is "frozen". In inflationary theory, time during inflation is often measured in "e-folds", where one e-fold is the amount of time over which the scale factor $a(t)$ increases by a factor of e. During inflation, when $a(t) \sim e^{Ht}$, this is the time interval Δt such that $e^{H(t+\Delta t)} = e^{Ht+1}$. Thus we relate R_k to the value of $\phi_k^p(x,t)$ evaluated a few e-folds after $t_{\text{exit}}(k)$.

This is the intuition behind the true relationship, which is given as [10]

$$R_k = -\frac{H}{\dot{\phi}_0(t_e)}\phi_k^p(x, t_e) \tag{3.18}$$

where t_e is a time a few e-folds after $t_{\text{exit}}(k)$. We can substitute this into Equation 3.15 to get a relationship between the width $\langle|\phi_k^p(t_e)|^2\rangle$ of the distribution for the kth mode of the perturbation field and the curvature power spectrum:

$$P_k = \frac{4\pi k^3}{V}\frac{H^2}{\dot{\phi}_0(t_e)^2}\left\langle|\phi_k^p(t_e)|^2\right\rangle \tag{3.19}$$

Finally, substituting this into Equation 3.16 gives us our key result: a relationship between the width of the kth mode of the perturbation field and the angular power spectrum:

$$C_\ell = \int T(k,\ell)^2\frac{4\pi k^2}{V}\frac{H^2}{\dot{\phi}_0(t_e)^2}\left\langle|\phi_k^p(t_e)|^2\right\rangle dk \tag{3.20}$$

This relationship is key to our result. If we can show that the stochastic modes ϕ_k^p have smaller widths at t_e than their quantum field theory counterparts for certain values of k, then we will know that the stochastic predictions for C_ℓ will be lower than the quantum field theoretical predictions.

3.3 Inflaton Field Modes

Now that we know the k-modes for a field operator, we can accomplish our main goal: explicitly describing the inflaton perturbation field, in its vacuum state, in terms of uncoupled modes. We will drop the p superscript for the perturbation field here and instead denote it simply as ϕ.

We start with the generic mode expansion of a field operator $\hat{\psi}$ in terms of the annihilation and creation operators $\hat{A}_k$, $\hat{A}_k^\dagger$, in which each mode is a solution to Equation 3.5:

$$\hat{\psi} = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2Vk^3}} \left(\left(\frac{k}{a} + iH \right) \hat{A}_k e^{i(k \cdot x + k/a)} + \left(\frac{k}{a} - iH \right) \hat{A}_k^\dagger e^{i(k \cdot x - k/a)} \right) \tag{3.21}$$

If the field ψ is in its lowest energy state, then every mode should correspond to a Gaussian probability distribution. From the previous section, we know that that the width of this distribution is directly related to the angular power spectrum. Thus our next two tasks will be to calculate the width of the mode ψ_k in quantum field theory and in stochastic quantum mechanics.

To get at the width of the distribution, we will first consider the two-point correlation function of the field in the vacuum at any given time. We choose to work with the Bunch-Davies vacuum for simplicity; this is the unique state in which a free-falling observer sees no particles [11].

$$\langle 0|\phi(x,t)\phi(x',t)|0\rangle \tag{3.22}$$

Since the vacuum sends annihilation operators to zero, the only surviving terms will be those with a creation operator on the right and an annihilation operator on the left (as a creation operator on the left would be equivalent to an annihilation operator on the right by Hermitian conjugation). Thus this two-point correlation will be of the form

$$\langle 0|\phi(x,t)\phi(x',t)|0\rangle = \sum_k \frac{\frac{k^2}{a^2} + H^2}{2Vk^3} e^{ik(x-x')} \tag{3.23}$$

We may calculate the width of the kth mode of ϕ by taking the Fourier transform of this two-point correlation function over all displacements:

$$\left\langle |\phi_k|^2 \right\rangle (t) = \frac{V}{(2\pi)^3} \int e^{-iky} \langle 0|\phi(x,t)\phi(x+y,t)|0\rangle d^3y \tag{3.24}$$

Substituting in the expression for the two-point correlation based on the vacuum annihilating the annihilator:

$$\left\langle |\phi_k|^2 \right\rangle (t) = \frac{V}{(2\pi)^3} \int e^{-iky} \sum_k \frac{\frac{k^2}{a^2} + H^2}{2Vk^3} e^{iky} d^3y \tag{3.25}$$

The exponentials cancel and extract one specific value of k:

$$\left\langle |\phi_k|^2 \right\rangle (t) = \frac{V}{(2\pi)^3} \int \frac{\frac{k^2}{a^2} + H^2}{2Vk^3} d^3y \tag{3.26}$$

The integral just multiplies this constant by the volume V, leaving us with

$$\left\langle |\phi_k|^2 \right\rangle (t) = \frac{V}{(2\pi)^3} \frac{\frac{k^2}{a^2} + H^2}{2k^3} \tag{3.27}$$

The term multiplied by the normalization constant is the width of the kth mode of ψ in its lowest-energy state:

$$\Delta_k^{2\,\mathrm{QFT}} = \left\langle |\phi_k(t_e)|^2 \right\rangle = \frac{\frac{k^2}{a^2} + H^2}{2k^3} \tag{3.28}$$

This is what the widths predicted by the stochastic theory will be compared to.

4 The Stochatic Inflaton Field

We will now convert the quantum field theory description of the inflaton field into a stochastic description. Note that every mode of the inflaton field is in its ground state, and thus corresponds to a harmonic oscillator. We can thus guess that the modes of the stochastic inflaton field are stochastic harmonic oscillators obeying the Ornstein-Uhlenbeck process:

$$d\phi_k = -v_+\phi_k dt + \sqrt{2D}dW \tag{4.1}$$

Consider an ensemble of particles that evolve with this differential equation. Dividing the distribution of the particles at a time t by the number of particles gives a probability density $p(x,t)$ for the position of any single particle in the distribution. This probability density in turn evolves according to the Fokker-Planck equation for this stochastic process:

$$\frac{\partial p}{\partial t} = v_+\frac{\partial}{\partial x}(xp) + D\frac{\partial^2 p}{\partial x} \tag{4.2}$$

The Green's functions of this equation for an initial distribution $p(x,t) = \delta(x_0, t_0)$ are Gaussians with widths given by

$$\Delta_k^2 = \frac{D}{v_+}\left(1 - e^{-2v_+(t-t_0)}\right) \tag{4.3}$$

In the long-time limit, the width of a stochastic mode ϕ_k must converge to the width of a classical mode. Thus the process that corresponds to the kth mode must satisfy

$$\frac{D}{v_+} = \frac{\frac{k^2}{a^2} + H^2}{2k^3} = \frac{1}{2k}\left(\frac{1}{a^2} + \frac{H^2}{k^2}\right) \tag{4.4}$$

This width is roughly proportional to $\frac{1}{k^3}$ when H is large and to $\frac{1}{k}$ when H is small. In both situations, the width (and thus the fraction D/v_+) is inversely proportional to k in some form.

We will then assume that the diffusion constant D is independent of k. This is analogous to the stochastic harmonic oscillator, as the wavenumber k is analogous to ω, while D contains information regarding Planck's constant $\hbar$ and the mass m of the oscillating particle. In order for the stochastic process to reproduce the ground state of the kth mode, this requires that v_+ be proportional to k.

Let $v_+ = Ck$ where C is includes the terms of the width that are not included in D. Then the width of the kth mode of the stochastic inflaton field at time t is given by

$$\left\langle |\phi_k(t)|^2 \right\rangle = \frac{1}{2k}\left(\frac{1}{a^2} + \frac{H^2}{k^2}\right)\left(1 - e^{-2Ck(t-t_0)}\right) \tag{4.5}$$

Letting $t_0 = 0$ and evaluating this width at a time t_e that's a few e-folds after $t_{\text{exit}}(k)$:

$$\left\langle |\phi_k(t_e)|^2 \right\rangle = \frac{1}{2k}\left(\frac{1}{a^2} + \frac{H^2}{k^2}\right)\left(1 - e^{-2Ckt_e}\right) \tag{4.6}$$

This is just the width predicted by quantum field theory with an extra factor of $\left(1 - e^{-2Ckt_e}\right)$. From this, we draw two significant conclusions:

1. Stochastic modes with lower k converge to equilibrium slower than modes with higher k.
2. At any time after t_0, low-k modes will have a smaller width (relative to equilibrium) than high-k modes.

Let us now consider the angular power spectrum C_ℓ for low ℓ. In this region, the transfer function $T(k,\ell)$ is dominated by the Sachs-Wolfe effect; that is, we can approximate

$$T(k,\ell)^2 \approx \pi H_0^4 j_\ell \left(\frac{2k}{H_0}\right)^2 \tag{4.7}$$

Under the assumption that the power spectrum P_k^{QFT} is roughly constant, we can then approximate C_ℓ^{QFT} as proportional to the experimentally-consistent $\frac{1}{2\ell(\ell+1)}$:

$$C_\ell^{QFT} \sim \int \frac{1}{k} j_\ell(k)dk = \frac{1}{2\ell(\ell+1)} \tag{4.8}$$

Thus the ratio $\frac{C_\ell}{C_\ell^{QFT}}$ is given by

$$\frac{C_\ell}{C_\ell^{QFT}} = \int \frac{1}{k} j_\ell \left(\frac{2k}{H_0}\right)^2 \frac{\left(1 - e^{-2Ckt_e}\right)}{\frac{1}{2\ell(\ell+1)}} dk \tag{4.9}$$

$$= 2\ell(\ell+1) \int \frac{1}{k} j_\ell \left(\frac{2k}{H_0}\right)^2 \left(1 - e^{-2Ckt_e}\right) dk \tag{4.10}$$

$$= 1 - 2\ell(\ell+1) \int \frac{1}{k} j_\ell \left(\frac{2k}{H_0}\right)^2 e^{-2Ckt_e} dk \tag{4.11}$$

The extent to which this ratio is less than 1 for each value of ℓ thus governs how much lower the angular power spectrum predicted by stochastic mechanics is than that predicted by quantum field theory. Note that this ratio is independent of the starting position x_0 of the stochastic harmonic oscillator; thus fixing of this parameter is not required.

5 Conclusion

We have suggested how a stochastic quantum theory can explain the anomalously low quadrupole moment in the temperature power spectrum of the CMB. The explanation is possible because the stochastic theory allows for non-equilibrium probablility distributions; that is, distributions which do not follow the Born rule. We started by describing the harmonic oscillator in stochastic mechanics and how it approaches equilibrium. We then described the stochastic inflaton field in its ground state by replacing each mode with a stochastic harmonic oscillator. Finally, we found that if the diffusion coefficient D of the harmonic oscillators is independent of k, the low-k modes of the field will approach equilibrium slower than the high-k modes. Thus when the modes are "frozen" by inflation, the ratio of non-equilibrium to equilibrium width will be smaller for low-k modes; thus the low-ℓ components of the temperature power spectrum will be smaller than in λCDM.

Future research might attempt to put this argument on firmer footing or expand the set of metrics on which it's applicable. One could derive the full form of D by deriving the Schrödinger equation from stochastic mechanics in an expanding metric, then finding the value of D that agrees with the classical solution. We see at least two ways to do this: follow the derivation given here in a curved metric, or attempt to derive the equations of motion from a variational principle akin to that of Yasue [12]. One could also consider the inflaton field in a curved metric, although this may be more complicated. Further research in the underlying theory of stochastic quantum gravity may give more insight into D and its dependence on k.

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