Chiral fermions and domain walls in lattice gauge theory

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by

Boyang Li

David Armstrong

Prof. David S. Armstrong

Christopher Monahan

Advisor: Prof. Christopher Monahan

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Abstract

Chiral domain wall fermions in lattice QCD have similar formulations to the physics of the quantum Hall effect. We study the correspondence between the existence of chiral edge modes in the presence of domain wall defects of Wilson fermions on a rectangular lattice and the Chern-Simons levels across the defects. We verify the computation of the Chern-Simons level of a Wilson fermion on a $2+1$ dimensional rectangular lattice with $1+1$ dimensional defects. We then generalize the calculations to a $4+1$ dimensional lattice and verify these procedures. We also present a naive attempt to compute the Chern-Simons level in $1+1$ dimensions and show that the results are incorrect as the Chern-Simons theory is not defined in even dimensions.

Chapter 1

Introduction

The goal of this paper is to study the correspondence between the existence of chiral edge modes in the presence of domain wall defect of Wilson fermions on a rectangular lattice and the Chern-Simons levels across the defects. In this chapter, we first introduce the key concepts relevant to our research and briefly go through the history of constructing chiral fermions on the lattice. Following Sen's work [1], in chapter 2 we verify the computation of the Chern-Simons level of a Wilson fermion on a $2+1$ dimensional rectangular lattice with $1+1$ dimensional defects. Then, we perform the same computations in $1+1$ dimensional rectangular lattices and show that this naive attempt yields trivial values of trace calculation, since the current Chern-Simons theory is not defined in even dimensional Euclidean spaces. Finally, we generalize these calculation procedures to 5 dimensional spaces and show that they correspond to our expected formula.

1.1 Terminologies

The study of fundamental particles and the Standard Model describing their relations has been of interests to physicists for decades. One reason is that the fundamental particles are the building blocks of everything in the visible universe, i.e., everything that exists is made up of them. Another reason is that the Standard Model incorporates the electromagnetic, strong and weak nuclear forces, three of the four fundamental forces of nature, and thus has become the most successful unifying theory so far. The study of particles boils down to the study of their quantum properties, such as mass, charge, spin or flavor. Among them, chirality, for its significant role in many quantum phenomena, has drawn much attention. The chirality of a particle is defined by whether it transforms in the right- or left-handed representation of the Poincaré group, whose unitary irreducible representations characterize different particles. For example, the chirality for a Dirac fermion ψ (a spin $\frac{1}{2}$ particle) is defined through the projection operators consisting of the operator γ^5, which projects the fermion field into its left- or right-handed component. A theory of Dirac fermions is said to have chiral symmetry if its Lagrangian is invariant under parity transformation (flip in the sign of all spatial coordinates). If a theory is asymmetric with respect to chiralities, it is called a chiral theory, and a vector theory if symmetric. For instance, the Electroweak theory, a unified description of two fundamental interactions of nature – electromagnetism and the weak interaction, is a chiral theory because only left-handed (whose directions of spin and motion are opposite) fermions or right-handed (whose directions of spin and motion are the same) antifermions (with the same mass, spin and mean lifetime but with charge, parity, strangeness and other quantum numbers flipped in sign) are observed to engage in the charged weak interaction. Quantum Chromodynamics (the theory of the strong interaction between quarks mediated by gluons), on the contrary, is a vector theory, since the coupling between quarks and gluons have no preference on chirality.

Next, we will introduce the lattice gauge theory. The term gauge actually refers to mathematical tricks that regulate redundant degrees of freedom in the Lagrangian of a physical system. A field theory becomes a gauge theory if its Lagrangian does not change under local transformations according to certain smooth families of operations

(Lie groups). The transformations between different gauges are called gauge transformations, and they form a Lie group, often referred to as the symmetry group or the gauge group of the theory. In continuous spacetime, calculations in non-perturbative gauge theory involve evaluating an infinite-dimensional path integral, which is computationally intractable. To solve this problem, we discretize the spacetime by working on lattices, and the path integral becomes finite-dimensional, which can be evaluated by stochastic simulation methods such as the Monte Carlo method.

Intuitively, when addressing multi-scale problems, we could only work with proper degrees of freedom that are relevant to the scales involved in our problem. For example, in quantum mechanics, we don't need to consider the influence of subatomic structures of particles to make a precise prediction of experiments. This is the main idea of Effective Field Theory [2]. In this research, it is explicitly demonstrated through the correspondence between the net chiralities of the Wilson fermion (high energy theory) across domain walls on a rectangular lattice and the Chern-Simons level (low energy theory) effective theory across the wall. This process involves integrating out a heavy Wilson fermion in the action.

Two adjoining structures or spaces are said to have Domain walls or topological defects at the boundaries where the joins are in some way "out of phase". Mathematically, it means that the topological solutions are homotopically distinct from vacuum solutions, i.e., the two functions cannot be continuously deformed into each other. For example, observe from Fig. 1.1 that the telephone wire is out of phase since the orientations are different at the knot.

1.2 History Review

Lattice QCD is a lattice gauge theory formulated on a discrete Euclidean space-time grid [3]. As a numerical method, Lattice QCD simulations are necessarily calculated

Figure 1.1: telephone wire

with finite lattice spacing. However, any non-zero lattice spacing a breaks the chiral symmetry of QCD, which can be recovered together with $O(4)$ symmetry in the limit $a \to 0$. Then, through Wick rotation, Lorentz symmetry can be recovered, leading to the correct target theory. Thus, the continuum construction of chiral fermions are of great interest. Wick rotation is a method of finding a construction in Minkowski space from a corresponding construction in Euclidean space, in a way that involves substituting a real-number variable by an imaginary-number variable. It is motivated by the observation that the Minkowski metric convention $(-1, +1, +1, +1)$ and the four-dimensional Euclidean metric are equivalent if the time coordinate is allowed to take imaginary values. As a non-perturbative approach to solving quantum chromodynamics (QCD), lattice QCD is so established that it serves as a framework for explaining many non-perturbative phenomena. However, simulating chiral fermions on the lattice had troubled physicists for twenty years, from the early 1970s to the early 1990s [4].

Naive attempts to put fermionic fields on a lattice in continuum theory result in the appearance of spurious states and an increase in the number of fermion species by 2^d, where d is the dimension of space-time, with a net chirality of zero. This is referred as the fermion doubling problem [5], caused by extra poles (remember that poles in the complex energy plane indicate the existence of particles) of the sine term in the

Dirac propagator when approximated around the Brillouin corners. Traditionally, it has been addressed using Wilson [6] or Kogut-Susskind [7] fermions, however at the expense of exact chiral symmetry at finite lattice spacing.

This problem was addressed in 1992 when Kaplan [8] proposed an alternative lattice fermion method, domain wall fermions (DWF), which adopted a continuum construction in lattice gauge theory. Kaplan showed that chiral fermions in $2n$ dimensions may be simulated by Dirac fermions in $2n+1$ dimensions with a domain wall defect in the mass parameter. Lattice regularization introduces doublers which leads to zero Chern-Simons level (defined in the next chapter). Since the original $2n+1$ dimensional theory is vector-like, the doublers could be removed by introducing a gauge invariant Wilson term, leading to a nontrivial Chern-Simons theory.

A later application [9] of the domain wall chiral fermion technique showed that the Chern-Simons level induced by Wilson fermions with a mass coupling to a domain wall on an odd-dimensional $d = 2n+1$ lattice can jump discontinuously at $d+1$ different values for the mass m, as a function of m/r, where r is the Wilson coupling constant. Commensurate changes in the number and chirality of zero modes on the domain wall occur with these jumps [10] in order to satisfy the anomaly inflow condition, which states that the calculation of chiral anomaly (nonconservation of a chiral current) cannot depend on the scale chosen for the calculation.

A recent paper [1] explored this phenomenon in detail in the context of a $2+1$ dimensional rectangular lattice with anisotropic lattice spacing. In particular, Sen showed 1): in the absence of a domain wall in the fermion mass, a $1+1$ dimensional defect can still exhibit chiral zero-mode solutions across which lattice spacing changes abruptly, and 2): on a uniform rectangular lattice, discrete changes in the number and chirality of zero modes occur on a domain wall in the fermion mass, as a function of lattice anisotropy.

Chapter 2

Chern-Simons Theory on a cubic lattice

In this section, we will first review the topological phases produced by a Wilson fermion on a cubic lattice coupled to a $U(1)$ lattice gauge theory, and then discuss how the results may change on a rectangular lattice.

2.1 The 3-Dimensional Case

Consider a heavy Wilson fermion of mass m and Wilson parameter r coupled to a $U(1)$ lattice gauge theory in $2+1$ dimensions. We start with the Dirac-Wilson operator on an infinite lattice with lattice spacing a_μ in the direction μ.

$$D_W = \sum_{\mu=1}^{3} \gamma_\mu \partial_\mu + m + \frac{r}{2} \sum_{\mu=1}^{3} \Delta_\mu \tag{2.1}$$

where ∂_μ is the lattice derivative $\partial_\mu = \frac{\delta_{z,z+a_\mu} - \delta_{z,z-a_\mu}}{2a_\mu}$ and Δ_μ is the lattice Laplacian $\Delta_\mu = \frac{\delta_{z,z+a_\mu} + \delta_{z,z-a_\mu} - 2\delta_{z,z}}{a_\mu^2}$

To understand how the topological transitions occur, consider first the Wilson fermion propagator given by

$$S^{-1}(p) = \sum_{\mu=1}^{d} i\gamma^\mu \frac{\sin(p_\mu a^\mu)}{a^\mu} + m + r \sum_{\mu=1}^{d} \frac{\cos(p_\mu a^\mu) - 1}{(a^\mu)^2}, \tag{2.2}$$

where $d = 3$ denotes the space-time dimension. In the long wavelength (low energy) limit for the weak gauge field, the Wilson fermion can be integrated out to arrive at the low energy effective theory for the $U(1)$ gauge field, which yields a Chern-Simons action for the gauge field: $S_{eff} = -i\frac{c}{4\pi}\Gamma_{C.S.}$ with

$$\Gamma_{C.S.} = \epsilon_{\alpha_1\beta_1\alpha_2} \int d^3x A_{\alpha_1} \partial_{\beta_1} A_{\alpha_2}. \tag{2.3}$$

The Chern-Simons level is an integer that characterizes the simple Lie group which specifies the low energy effective theory, i.e., the Chern-Simons theory. It is denoted by the constant 'c' and can be calculated from the Feynman diagram (Fig. 2.1) as

$$c = -\frac{4\pi\epsilon_{\alpha_1\beta_1\alpha_2}}{2(3!)} \partial_{(q_1)_{\beta_1}} \int_{\mathrm{BZ}} \frac{d^3p}{(2\pi)^3} \mathrm{Tr}(S(p)\Lambda_{\alpha_1}(p, p-q_1)S(p-q_1)\Lambda_{\alpha_2}(p+q_2, p)) \,|_{q_i=0} \tag{2.4}$$

where gauge invariance implies that the photon coupling satisfies the Ward identity

$$\Lambda_\mu(p, p) = -i\partial_{p_\mu} S^{-1}(p). \tag{2.5}$$

Then, the Chern-Simons level c can be reformulated using fermion propagator as

$$c = \frac{\epsilon_{\mu_1\mu_2\mu_3}}{2(3!)} \int_{\mathrm{BZ}} \frac{d^3p}{(2\pi)^3} \mathrm{Tr}([S(p)\partial_{p_{\mu_1}} S^{-1}(p)][S(p)\partial_{p_{\mu_2}} S^{-1}(p)][S(p)\partial_{p_{\mu_3}} S^{-1}(p)]). \tag{2.6}$$

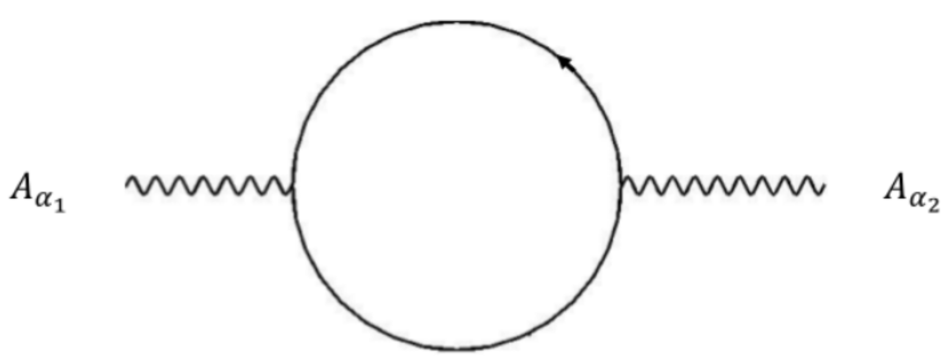

Figure 2.1: The one-loop feynman diagram producing the Chern-Simons level.

We begin by calculating the trace in Eq. (2.6). First note that

$$\partial_{p_\mu} S^{-1}(p) = i\gamma^\mu \cos(p_\mu a^\mu) - r\frac{\sin(p_\mu a^\mu)}{a^\mu}.$$

Now, let's compute $S(p)$ by taking

$$M = m + r\sum_{\mu=1}^{d} \frac{\cos(p_\mu a^\mu) - 1}{(a^\mu)^2}.$$

Then,

$$\begin{aligned} S(p) &= \frac{1}{\sum\limits_{\mu=1}^{d} i\gamma^\mu \frac{\sin(p_\mu a^\mu)}{a^\mu} + M\mathbb{1}} \\ &= \frac{\sum\limits_{\mu=1}^{d} -i\gamma^\mu \frac{\sin(p_\mu a^\mu)}{a^\mu} + M\mathbb{1}}{(\sum\limits_{\mu=1}^{d} i\gamma^\mu \frac{\sin(p_\mu a^\mu)}{a^\mu} + M\mathbb{1})(\sum\limits_{\nu=1}^{d} -i\gamma^\nu \frac{\sin(p_\nu a^\nu)}{a^\nu} + M\mathbb{1})} \\ &= \frac{\sum\limits_{\mu=1}^{d} -i\gamma^\mu \frac{\sin(p_\mu a^\mu)}{a^\mu} + M\mathbb{1}}{\sum\limits_{\mu=1}^{d} \frac{\sin^2(p_\mu a^\mu)}{(a^\mu)^2} + M^2\mathbb{1}}. \end{aligned} \tag{2.7}$$

Since we are in Euclidean space, we don't have Einstein's summation convention. However, we can define a similar notation without causing problems. Denote $\not{p} = \gamma^\mu p_\mu$ to be the sum in equation (2), where $p_\mu = \frac{\sin(p_\mu a^\mu)}{a^\mu}$ and $p^2 = p_1^2 + p_2^2 + p_3^2 = \sum\limits_{\mu=1}^{d} \frac{\sin^2(p_\mu a^\mu)}{(a^\mu)^2}$. Then, $S(p)$ can be rewritten as

$$S(p) = \frac{-i\not{p} + M\mathbb{1}}{p^2 + M^2}. \tag{2.8}$$

To compute the Chern-Simons level c from Eq. (2.6), we expand the momentum space integral near the Brillouin zone (BZ) corners. We first examine the case where all components of the momenta equal to 0. Taking the approximation first, the Wilson fermion propagator becomes

$$S^{-1}(p) = m + \sum_{\mu=1}^{d} i\gamma^\mu p_\mu. \tag{2.9}$$

We then have $\partial_{p_\mu} S^{-1}(p) = i\gamma^\mu$ and

$$S(p) = \frac{-i\not{p} + m}{p^2 + m^2}. \tag{2.10}$$

Finally, let's consider the trace. At first, we performed the calculations using the Mathematica package FeynCalc, but the results didn't fit our expectation. We later discovered that this was because FeynCalc assumed the calculations were done in 4-d Minkowski spaces, so it assumed there were 4 gamma matrices. Having no convenient package to use, we decided to define gamma matrices explicitly. Since we are in the 3-d Euclidean space, they can be defined as Pauli matrices.

Now, let

$$H = [S(p)\partial_{p_{\mu_1}} S^{-1}(p)][S(p)\partial_{p_{\mu_2}} S^{-1}(p)][S(p)\partial_{p_{\mu_3}} S^{-1}(p)],$$

and consider $\mu_i = i$, $i \in \{1, 2, 3\}$. Plugging in everything, we have

$$H = \frac{-i(-i\not{p} + m)\sigma_{\mu_1}(-i\not{p} + m)\sigma_{\mu_2}(-i\not{p} + m)\sigma_{\mu_3}}{(p^2 + m^2)^3}. \tag{2.11}$$

Now we are ready to perform the trace calculations in Eq. (2.6). Taking into account the Levi-Civita symbol $\epsilon_{\mu_1\mu_2\mu_3}$, there are only six combinations of indexes that yield no-zero values. Mathematica gives the following result (Fig. 2.2). The C.S. is expected to be

$$c = \sum_{k,\alpha} \int_{d\Omega} (-1)^k \frac{d^3p}{2\pi^2} \frac{(m - 2\frac{r}{a_l^2 k})}{(p^2 + (m - 2\frac{r}{a_l^2 k})^2)^2}, \tag{2.12}$$

where k stands for the number of components of the momenta equal to π while the rest equal to 0, and $\alpha = 1, \cdots, \binom{d}{k}$ counts for the permutations. We can see that our calculation meets the expectation, taking $k = 0$ in Eq. (2.12).

To verify our computation procedures, we also consider the $k = 1$ case. Since

```
In[31]:= Simplify[Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ1.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ2.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ3] -
          Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ1.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ3.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ2] -
          Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ2.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ1.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ3] +
          Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ2.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ3.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ1] +
          Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ3.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ1.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ2] -
          Tr[-I*(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ3.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2])
            .σ2.(-I*(σ1*p1+σ2*p2+σ3*p3)+m*IdentityMatrix[2]).σ1]]

Out[31]= 12 m (m^2 + p1^2 + p2^2 + p3^2)
```

Figure 2.2: 3d C.S. level at k=0

$\sin x = -x$ expanded at $x = \pi$, the propagator and trace become

$$
\begin{aligned}
S^{-1}(p) &= M + \sum_{\mu=1}^{2} i\gamma^{\mu} p_{\mu} - i\gamma^{3}, \\
S(p) &= \frac{-i\not{p} + M}{p^2 + M^2}, \\
H &= \frac{i(-i\not{p} + M)\sigma_{\mu_1}(-i\not{p} + M)\sigma_{\mu_2}(-i\not{p} + M)\sigma_{\mu_3}}{(p^2 + M^2)^3},
\end{aligned}
$$

in the case where only the third coordinate is non-zero and equal to π. Note that $M = m - 2\frac{r}{a_l^2 k}$ and $\not{p} = \sum_{\mu=1}^{2} i\gamma^{\mu} p_{\mu} - i\gamma^{3}$. Mathematica yields the following result (Fig. 2.3), which is the same as taking $k = 1$ in Eq. (2.12). Following the similar procedures, the cases $k = 2, k = 3$ are verified.

2.2 The 2-Dimensional Case

One might think that the generalization of these calculations to arbitrary dimension is straightforward: just change d and find the corresponding gamma matrices satisfying the Clifford algebra anti-commutation relation. However, this naive attempt is incor-

```
In[35]:= Simplify[Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ1.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ2.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ3] -
    Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ1.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ3.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ2] -
    Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ2.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ1.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ3] +
    Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ2.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ3.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ1] +
    Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ3.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ1.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ2] -
    Tr[I * (-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ3.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2])
      .σ2.(-I * (σ1 * p1 + σ2 * p2 - σ3 * p3) + m * IdentityMatrix[2]).σ1]]
```

Out[35]= $-12\, m\, (m^2 + \mathrm{p1}^2 + \mathrm{p2}^2 + \mathrm{p3}^2)$

Figure 2.3: 3d C.S. level at k=1

rect for the following reason. Kaplan's original lattice chiral fermion method [9] was formulated in odd dimensional $(2n+1)$ lattice theory, and Witten's foundational work in 1980's on Wilson loops in Chern-Simons theory was also based on odd-dimensional spacetime [11]. In his formulation of the C.S. theory, the dimension of spacetime must be odd [12], for reasons from differential geometry and Lie algebra [13]. Theoretically, one could introduce a new definition of C.S. theory in even dimensions, though there are very few studies of this to date. The current formulations employ knowledge from non-commutative geometry and Riemannian spin maniforlds, which are beyond the scope of this work, so I will just share the references for readers' interests [14, 15].

In this section, I will naively set $d = 2$ and perform the similar calculations from the previous section to show how this attempt could be incorrect. Through these analyses, we hope to shine light on how future work could be done to construct a 2-d C.S. theory, which could be very useful since it should be easier to simulate numerically.

We begin by identifying the gamma matrices in $1+1$ dimensions. First, observe that $Cl(0,2) \cong \mathbb{R}(2)$, where $\mathbb{R}(2)$ is the algebra of 2×2 real matrices. It turns out that there is a unique two-dimensional irreducible representation of $Cl(0,2)$, and the two gamma matrices can be identified as

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Following similar procedures and notations, the formula to the compute the C.S. level is given by

$$c = \frac{\epsilon_{\mu_1\mu_2}}{2(2!)} \int_{\mathrm{BZ}} \frac{d^3p}{(2\pi)^3} \mathrm{Tr}([S(p)\partial_{p_{\mu_1}} S^{-1}(p)][S(p)\partial_{p_{\mu_2}} S^{-1}(p)].$$

where is the 2-d Levi-Civita symbol defined as

$$\epsilon_{\mu_1\mu_2} = \begin{cases} +1, & \text{if } (i,j) = (1,2) \\ -1, & \text{if } (i,j) = (2,1) \\ 0, & \text{if } i = j. \end{cases}$$

Then, the trace H can be written as

$$H = \frac{-i(-i\not{p} + m)\sigma_{\mu_1}(-i\not{p} + m)\sigma_{\mu_2}}{(p^2 + m^2)^3},$$
$$H = \frac{(-i\not{p} + M)\sigma_{\mu_1}(-i\not{p} + M)\sigma_{\mu_2}}{(p^2 + M^2)^3},$$

respectively for $k = 0, 1$.

The trace calculation using Mathematica yields trivial results for both cases (Fig. 2.4). Further examinations show that the trace still equals zero for other BZ corner approximations. This may suggest that we are using the wrong formula, and our naive attempt fails.

2.3 The 5-Dimensional Case

According to Kaplan [8], in odd dimensions $2n + 1$ the C.S. level is defined by

$$c = \frac{(-i)^n \epsilon_{\mu_1 \cdots \mu_{2n+1}}}{(n+1)(2n+1)!} \int_{\mathrm{BZ}} \frac{d^{2n+1}p}{(2\pi)^{2n+1}} \mathrm{Tr}([S(p)\partial_{p_{\mu_1}} S^{-1}(p)] \cdots [S(p)\partial_{p_{\mu_{2n+1}}} S^{-1}(p)]). \tag{2.13}$$

```
Tr[-(-I*(γ1*p1+γ2*p2)+m*IdentityMatrix[2]).γ1.(-I*(γ1*p1+γ2*p2)+m*IdentityMatrix[2])
 迹     虚数单位              单位矩阵                 虚数单位              单位矩阵
   .γ2] - Tr[-(-I*(γ1*p1+γ2*p2)+m*IdentityMatrix[2]).γ2.(-I*(γ1*p1+γ2*p2)+m*IdentityMatrix[2])
          迹     虚数单位              单位矩阵                 虚数单位              单位矩阵
   .γ1]

Out[47]= i p2 (-m - i p1) + i p2 (m + i p1)

In[48]:= Simplify[%]
         化简

Out[48]= 0

In[49]:= Tr[(-I*(γ1*p1-γ2*p2)+m*IdentityMatrix[2]).γ1.(-I*(γ1*p1-γ2*p2)+m*IdentityMatrix[2])
         迹    虚数单位              单位矩阵                 虚数单位              单位矩阵
   .γ2] - Tr[(-I*(γ1*p1-γ2*p2)+m*IdentityMatrix[2]).γ2.(-I*(γ1*p1-γ2*p2)+m*IdentityMatrix[2])
          迹    虚数单位              单位矩阵                 虚数单位              单位矩阵
   .γ1]

Out[49]= i p2 (-m - i p1) + i p2 (m + i p1)

In[50]:= Simplify[%]
         化简

Out[50]= 0
```

Figure 2.4: 2d C.S. levels at k=0,1

In 5-d, the Levi-Civita symbol $\epsilon_{\mu_1\cdots\mu_5}$ is given by [16]

$$\epsilon_{\mu_1\cdots\mu_5}=\frac{1}{288}(\mu_2-\mu_1)(\mu_3-\mu_1)(\mu_4-\mu_1)(\mu_5-\mu_1)(\mu_3-\mu_2)(\mu_4-\mu_2)(\mu_5-\mu_2)(\mu_4-\mu_3)(\mu_5-\mu_3)(\mu_5-\mu_4),$$

and the gamma matrices can be defined as

$$\gamma_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\gamma_2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},\gamma_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$
$$\gamma_4=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},\gamma_5=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

We first examine the case where all components of the momenta equal to 0. Taking the approximation first, recall that the Wilson fermion propagator and its derivative become

$$
\begin{aligned}
S^{-1}(p) &= m + \sum_{\mu=1}^{5} i\gamma^{\mu} p_{\mu}, \\
S(p) &= \frac{-i\not{p} + m}{p^2 + m^2}, \\
\partial_{p_\mu} S^{-1}(p) &= i\gamma^{\mu}, \\
H &= \frac{i \prod_{j=1}^{5} (-i\not{p} + m)\gamma_{\mu_j}}{(p^2 + m^2)^5}.
\end{aligned}
$$

The trace calculation (taking into account the permutations) using Mathematica gives the result $480m\left(m^2 + p^2\right)^2$. Now consider the case where only 1 momentum component is not 0, say p_5. Following the same notations as before, we have

$$
\begin{aligned}
S^{-1}(p) &= M + \sum_{\mu=1}^{4} i\gamma^{\mu} p_{\mu} - i\gamma^{5}, \\
S(p) &= \frac{-i\not{p} + M}{p^2 + M^2}, \\
H &= \frac{-i \prod_{j=1}^{5} (-i\not{p} + M)\gamma_{\mu_j}}{(p^2 + M^2)^5}.
\end{aligned}
$$

Mathematica gives the result $-480m\left(m^2 + p^2\right)^2$. Following the same procedures, the cases $k = 1$ up to 5 are verified. We attach the codes in appendix. However, there's one thing worth noticing: although the form of our calculations of C.S. level fit the expectation (Eq. (2.12)), it differs by a constant factor $\frac{480}{(2+1)(4+1)!} = \frac{480}{360}$. This may lead to non-integer value of the C.S. level after plugging in the lattice anisotropy $\frac{a_s}{a}$ and the Wilson parameter $\frac{r}{a_s}$, which are forbidden in the Chern-Simons theory. We leave this analysis to future work.

Chapter 3

Conclusion and Future Work

In this paper, we give an overview of the attempts to put fermionic fields on a lattice and then introduce the construction of chiral fermions in lattice QCD. We explicitly calculate the Chern-Simons level in 3 dimensional Euclidean space, verifying Sen's results, and generalize these procedures to 5 dimensions, building on Kaplan's work. However, the construction of Chern-Simons theory in even dimensional Euclidean space fails in this formulation, and entails new definitions using knowledge from non-commutative geometry and Riemannian spin maniforlds. We show this by naively changing the dimension $d = 3$ to 2 and defining the corresponding gamma matrices that satisfy the Clifford algebra anti-commutation relation. Trivial results of the trace calculations at all Brillouin corners approximations indicate the construction is ill-defined. In future, if a 2 dimensional construction were established, it would become a very useful toy model for us to understand the theory, as lattice QCD simulations on computer would be fast. Finally, although the form of our calculations of Chern-Simons level in 5-d fit the expectation, it differs by a constant factor. This may lead to non-integer value of the Chern-Simons level after plugging in the lattice anisotropy $\frac{a_s}{a}$ and the Wilson parameter $\frac{r}{a_s}$, which are forbidden in the Chern-Simons theory. Once future work yields correct values of the Chern-Simons level, we are curious to see outcome of lattice QCD calculations, because the domain wall becomes a 4 dimensional space, on which we have many established results.

Appendix A

Mathematica codes for the 5-d trace calculations

Define Levi-Civita Symbol in 5d

```
In[1]:= eps[i1_, i2_, i3_, i4_, i5_] := 1 / 288 * (i2 - i1) (i3 - i1)
          (i4 - i1) (i5 - i1) (i3 - i2) (i4 - i2) (i5 - i2) (i4 - i3) (i5 - i3) (i5 - i4)

In[2]:= eps[1, 2, 3, 4, 2]

Out[2]= 0

In[3]:= temp = 0;
        Do[If[eps[i, j, k, m, n] == 0, temp = temp + 1], {i, 5}, {j, 5}, {k, 5}, {m, 5}, {n, 5}];
        ⌊… ⌊如果
        Print[temp + 120 == 5^5]
        ⌊打印
        True
```

Define Gamma matrices in 5d

```
In[4]:= γ1 = {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}

Out[4]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}

In[5]:= γ2 = {{0, 0, 0, -I}, {0, 0, I, 0}, {0, -I, 0, 0}, {I, 0, 0, 0}}
                        ⌊虚数单位    ⌊虚数单位      ⌊虚数单位    ⌊虚数单位

Out[5]= {{0, 0, 0, -i}, {0, 0, i, 0}, {0, -i, 0, 0}, {i, 0, 0, 0}}

In[6]:= γ3 = {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}}

Out[6]= {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}}

In[7]:= γ4 = {{0, 0, -I, 0}, {0, 0, 0, -I}, {I, 0, 0, 0}, {0, I, 0, 0}}
                    ⌊虚数单位          ⌊…   ⌊虚数单位          ⌊虚数单位

Out[7]= {{0, 0, -i, 0}, {0, 0, 0, -i}, {i, 0, 0, 0}, {0, i, 0, 0}}

In[8]:= γ5 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}

Out[8]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}

In[9]:= γ = {γ1, γ2, γ3, γ4, γ5}

Out[9]= {{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
         {{0, 0, 0, -i}, {0, 0, i, 0}, {0, -i, 0, 0}, {i, 0, 0, 0}},
         {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
         {{0, 0, -i, 0}, {0, 0, 0, -i}, {i, 0, 0, 0}, {0, i, 0, 0}},
         {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}}

In[10]:= γ[[1]].γ[[2]] + γ[[3]].γ[[4]]

Out[10]= {{2 i, 0, 0, 0}, {0, -2 i, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[11]:= For[i = 1, i < 6, i++, For[j = 1, j < 6, j++, x = γ[[i]].γ[[j]] + γ[[j]].γ[[i]];
         ⌊For循环                 ⌊For循环
           Print[x == 2 * KroneckerDelta[i, j] * IdentityMatrix[4]]]]
           ⌊打印            ⌊克罗内克尔δ函数              ⌊单位矩阵
```

```
True
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True
```

Perform the Trace calculation (all 0 momentum)

```
Sp = -I * (γ[[1]] * p1 + γ[[2]] * p2 + γ[[3]] * p3 + γ[[4]] * p4 + γ[[5]] * p5) + m * IdentityMatrix[4]
          虚数单位                                                                   单位矩阵
```

```
Out[•]= {{m - ⅈ p5, 0, -ⅈ (p3 - ⅈ p4), -ⅈ (p1 - ⅈ p2)}, {0, m - ⅈ p5, -ⅈ (p1 + ⅈ p2), -ⅈ (-p3 - ⅈ p4)},
  {-ⅈ (p3 + ⅈ p4), -ⅈ (p1 - ⅈ p2), m + ⅈ p5, 0}, {-ⅈ (p1 + ⅈ p2), -ⅈ (-p3 + ⅈ p4), 0, m + ⅈ p5}}
```

the derivative equal i*γj, see thesis pg.7 eq 2.10.

```
In[36]:= temp = 0; Do[temp =
                    Do循环
  temp + I^5 * eps[i, j, k, p, q] × Tr[I * Sp.γ[[i]].Sp.γ[[j]].Sp.γ[[k]].Sp.γ[[p]].Sp.γ[[q]]],
           虚数单位                   迹  虚数单位
 {i, 5}, {j, 5}, {k, 5}, {p, 5}, {q, 5}]
```

```
In[37]:= Simplify[temp]
         化简
```

Out[37]= $480\, m \left(m^2 + p1^2 + p2^2 + p3^2 + p4^2 + p5^2\right)^2$

Perform the Trace calculation (1 momentum neq 0)

In[43]:=
```
Sp = -I * (γ[[1]] * p1 + γ[[2]] * p2 + γ[[3]] * p3 + γ[[4]] * p4 - γ[[5]] * p5) + m * IdentityMatrix[4]
```
虚数单位 单位矩阵

Out[43]=
```
{{m + i p5, 0, -i (p3 - i p4), -i (p1 - i p2)}, {0, m + i p5, -i (p1 + i p2), -i (-p3 - i p4)},
 {-i (p3 + i p4), -i (p1 - i p2), m - i p5, 0}, {-i (p1 + i p2), -i (-p3 + i p4), 0, m - i p5}}
```

In[47]:=
```
temp = 0; Do[temp =
  temp + I^5 * eps[i, j, k, p, q] × Tr[-I * Sp.γ[[i]].Sp.γ[[j]].Sp.γ[[k]].Sp.γ[[p]].Sp.γ[[q]]],
 {i, 5}, {j, 5}, {k, 5}, {p, 5}, {q, 5}]
```
Do循环 虚数单位 迹 虚数单位

In[48]:=
```
Simplify[temp]
```
化简

Out[48]= $-480\, m \left(m^2 + p1^2 + p2^2 + p3^2 + p4^2 + p5^2\right)^2$

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