Taming of Monsters: Expansion of the Applications of Fractal Geometry

A thesis submitted in partial fulfillment of the requirements for the Bachelor of Science degree in Physics from the College of William & Mary

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Because my own words fail me,

Bohr: Before we can lay our hands on anything, our life's over.
Heisenberg: Before we can glimpse who or what we are, we're gone and laid to dust.
Bohr: Settled among all the dust we raised.

-Michael Frayn, *Copenhagen* (93)

In loving memory of Paul Daniel Soutter.

Thank you for reminding us to find order in the chaos.

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Introduction

Michael Frayn[1], said to be "the giant who bestrides the British arts,"[2] briefly discusses fractals in his book of philosophy, *The Human Touch*. Much of the book discusses the inherent limitations of our observations from our human perspective when the smallest component of any given object is breaks down to nothing more realizable than indeterminate probabilities and wave formations. He details the human efforts to impose constructions upon a chaotic universe for the purposes of understanding and correlating observations, noting the successes and pitfalls of such approximations. Fractals, he says, are "an admirable attempt to come to terms with the irregularity of the world – the irregularity in this case being a function not of the subjective indeterminacy imposed by human observation, but of an objective feature of the universe…"[3] We will come to see how fractal geometry is a human construct, a conceptual shell imposed on observable irregular forms, but the necessity of this understanding is brought about by concrete, objective features of the universe.

These irregular structures that necessitate a fractal understanding of the world "were regarded…as a 'gallery of monsters,' kin to the cubist painting and atonal music that were upsetting the established standards of taste in the arts at about the same time."[4] During the 20th century we saw a development in both the sciences and the humanities to assimilate understanding of chaos into the standard image of the world. This expansion of the human understanding into chaotic forms is integral to our understanding of how fractals both naturally and purposefully exist and what the trends towards either

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[1] My concurrent project is a discussion of Frayn's works, focusing on his scientific historical fiction play *Copenhagen*. The significance of this will be further discussed in our discussion of art.
[2] "There's Still Life in the Old Stager; Profile." *Sunday Times.* (London, UK): 15. 2002.
[3] Frayn, Michael. *The Human Touch*. (New York: Metropolitan, 2006), 104.
[4] Dyson, Freeman. "Characterizing Irregularity." *Science*: 200. May 12, 1978. 677-678.

probabilistic naturalism or strict mathematical perfection mean about the nature of the element in question.

Background

Fractals are something of a murky backwater of arithmetic, the applications of which are omnipresent. To most of the population, fractals are nothing more than a pretty desktop background or an elevated word to describe what one sees in a kaleidoscope, and geometry goes no further than eighth grade trigonometric ratios and how to bisect a line with a compass. The true treatment of fractal math seems to live in this unnecessarily esoteric realm for how prevalent the forms are in natural systems. Euclidean and Newtonian approximations of complex and irregular mathematical and physical forms dominate conventional interpretation for the sake of simplicity. But filing objects into ill-fitting categories and seeing the world through classical eyes will only yield so much. "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."[5] With a fuller understanding of the concepts and applications of fractality, they describe all manner of natural and theoretical systems that other mathematical interpretations of forms dismiss as amorphous or nonstructured.

Symmetry and similarity are classifications of forms that we are intimately familiar with. The standard Euclidean symmetry we learn comes in three main forms: reflectional, rotational, and translational. If a form is reflected across a certain given axis, rotated by a discrete angle, or translated through space, it will appear as the same form. Another exhibition of symmetry arises when we speak of scale-invariance. The term is fairly self-explanatory – that the object or system in question does not vary or change as

[5] Mandelbrot, Benoit B. *The Fractal Geometry of Nature*. (San Francisco: W.H. Freeman, 1982), 1

the scale changes. In other words, scale-invariant or self-similar objects appear the same at all scales.

However, this self-similarity is not the only qualification to classify an object or system as "fractal." In a fractal we must see self-similarity throughout the structure, not just around a single point. To illustrate this, compare a spiral to a snowflake. A spiral will appear the same if magnified, but only around its center; a snowflake will have complexity that may appear self-similar independent of where one varies the scale.

Fractals are what could be called a "soft concept" in math. The term "fractal" describes objects, shapes, quantities, etc. that display a self-similar construction on all scales. In the purest mathematical sense, this self-similarity would be exact on all scales; however, the coining of the term by Benoit Mandelbrot was intended as slightly more of a metaphoric concept to describe a phenomenon. In fact, in *The Fractal Geometry of Nature*, Mandelbrot says that "this work pursues neither abstraction nor generality for its own sake, and is neither a textbook nor a treatise in mathematics…it is written from a personal point of view and without attempting completeness"[6]. The father of fractal geometry thought of it as less of a hard, closed subject, and more an exploration of a new form of geometry altogether. Euclidian geometry fails when it attempts to discuss fractal forms, and would rather refer to them as amorphous, despite there being distinctive organization and pattern to them. This alternative geometry fills that void and gives order to the former unclassifiable chaos.

He delineated between concept and manifestation with qualifiers attached to the term fractal. "The combination *fractal set* will be defined rigorously, but the combination *natural fractal* will serve loosely to designate a natural pattern that is usefully

[6] Mandelbrot. *The Fractal Geometry of Nature*. (2)

representable by a fractal set. For example, Brownian curves are fractal sets, and physical Brownian motion is a natural fractal."[7] Conceptual mathematical fractals conform perfectly to an established set of rules for the form, while natural fractals are more rough fittings of these conceptual forms in observable systems. For the scope of this paper, we are primarily interested in the difference between these two applications of the same term, where it arises, and if and how it is useful to use the same term with qualifiers to describe different systems.

It would be a fallacy to attempt to examine fractals purely from a cold, mathematical perspective. They were born of nature and nature is inherent in them. Nature is chaotic and difficult to classify perfectly, thus we must create and constantly redefine a dialect with which we can discuss and understand. As Galileo said, "[the universe] is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it…"[8] Approaching the universe with a limited vocabulary of Euclidean forms and Newtonian motion is not sufficient to encompass the full picture. Likewise, examining purely scientific applications limits the world available and the language capable to express fractality in its myriad forms.

As discussed earlier, Mandelbrot's work was part of a larger intellectual revolution in the 20th century. In mathematics, we see this revolution against strictly Euclidean interpretation towards a more complex understanding of fractals. In physics we see the same movement away from strictly Newtonian interpretations of the workings of the natural world towards quantum mechanics and chaos theory. The revolution

[7] Mandelbrot. *The Fractal Geometry of Nature*. (4-5)
[8] Popkin, Richard Henry. *A Philosophy of the Sixteenth and Seventeenth Centuries*. (Free Press: 1966), 65

extends further, even into the artistic realm. In visual art, we see symbolism, expressionism, cubism, and Dadaism. In music we see atonal and microtonal compositions, experiments with form and tonality with the modernists, and minimalism take shape. In theatre, we find reactions against the naturalistic tradition in symbolism, theatre of cruelty, epic theatre, and absurdism. We will discuss the full implications of these developments in greater detail later on, but the common theme in all of these shifts is reconciliation with the form of chaos; taming the monsters. Dane Camp extrapolates upon Galileo's quote, saying in his review of Mandelbrot's life and work,

> "Mathematics is a language, the language of the universe. Students who want to appreciate the poetry of the cosmos need to learn the vocabulary, grammar, and structure of the language. They must become articulate if they are to apply the language to practical discourse. Also, if they want to understand the interconnectedness of the universe, they must be acquainted with a wide variety of its literature."[9]

In the same way mathematics is the language of the universe, language itself, our usage of it, and artistic expression are the way in which we express our understanding of the universe. We impose form and constructions upon the world and by doing so develop our understanding. Our understanding of the world is in turn expressed through forms and constructions that we create. The continual two-way informative, influential process between comprehension and creation makes understood forms in both synthesis and analysis perpetually relevant in all fields of both exploration and expression.

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[9] Camp, Dane. "Benoit Mandelbrot: The Euclid of Geometry." *The Mathematics Teacher*: 93, 8. Nov, 2000. (708-712)

This discussion of fractal geometry has no pretense of being a rigorous mathematical or scientific analysis. Rather, we shall examine instances of application, demonstrating the advantages of the shift of perspective from looking at objects and systems through a fractal lens and the utility of the perspective outside of the realm of scientific and mathematical analysis. We have seen that a fractal set is a perfect, infinite representation of a form or pattern that abides by preset rules according to the human construction of a "fractal." It is a purely conceptual creation that abides by a definition. Natural fractals, conversely, are stochastic and imperfect. Through the discussion of fractals exhibited in nature, man-made materials, and art, the tendency for materials to exhibit a natural, stochastic formation striving towards the perfect fractal set and for art to largely exist as a suggestion of the fractal set striving towards the natural (depending on the intended message) arises. Because of the multifaceted applications of the concept pertaining to both the fractal set and the natural fractal for all manner of disciplines, a universal dialect of fractality is necessary.

In Nature

I will begin our more detailed discussion of fractals with examples in nature, as it is out of this necessity to describe complex natural forms that the concept of a fractal was born. The treatment of natural fractals and fractal forms in man-made materials should be fairly cursory and easy to manage. Both have undergone rigorous analysis in the canon of fractal geometry and are therefore both well understood to exhibit fractal qualities. The application of the term is largely agreed to be appropriate. Art will take slightly more discussion, as it is a more unorthodox application of fractal understanding, but we will reach that later on. It is clear that we see representatives of fractals in nature. They are exhibited in places we would not even expect with knowing the form and structure of a geometric fractal. From trees to mountain faces to measuring coastlines to the stock market, fractals can be used to describe a litany of natural systems.

As I quoted from Mandelbrot's book in the introduction, there are many natural forms that cannot be accurately approximated in Euclidean terms. To attempt to describe nature purely in terms of Euclidian geometry would be a mistake, as nature displays not only a higher level of complexity than described in Euclidian terms, but altogether a different kind of complexity. In these cases, a dialectical shift to the language of fractals becomes immensely useful to us to describe previously immensely difficult systems.

Natural fractals and what Mandelbrot calls fractal sets differ remarkably. The fractal concept itself remains – that at any scale the structure appears the same; parts resemble the whole. However, compare a mathematically constructed fractal tree to a natural tree. The two look qualitatively similar, but nature does not conform to mathematical perfection. Different features such as resource availability, anisotropic field influences, mechanical strength of the system at an upper boundary, etc. will divert the growth of a fractal phenomenon away from mathematical perfection. Thus, rather than the system being indistinguishable at any scale, we see a "rough" self-similarity. Statistical self-similarity manifests, meaning that the *distribution* of features is self-similar, rather than it perfectly aligning. Here, with this statistical self-similarity and stochasticism, is where we introduce the concept of chance into fractality. Natural fractals progress stochastically, which is to say that they progress from generation to generation with some probability distribution of perfect self-similarity.

There has been plenty of opposition to the idea of fractals, beginning when Mandelbrot first proposed the idea. A detailed study published in *Science* seems to be in direct response to Mandelbrot's assertions. The title "Is the Geometry of Nature Fractal?"[10] particularly throws the gauntlet in the direct questioning of the title of Mandelbrot's book. This article does not refute fractals as a mathematical concept, rather, it praises them as "beautiful mathematical constructs."[11] The contention made is that the term 'fractal' is abused in reports of experimental analysis of natural forms. They reviewed all experimental papers reporting a fractal analysis over the course of seven years in all *Physical Review* journals (a total of 96 experimental papers).

> "In these papers, an empirical fractal dimension D was calculated from various relations between a property P and the resolution r of the general form $P=kr^{f(D)}$, where k is the prefactor for the power law and the exponent $f(D)$ is a simple function of D."[12]

[10] Anir, David; Biham, Ofer; Lidar, Daniel; Malcai, Ofer. "Is the Geometry of Nature Fractal?" *Science*: Jan 2, 1998; 279, 5347. (39-40)
[11] Anir, David, *et al*. IBID. 39
[12] Anir, David, *et al.* IBID. 39

The scaling range distribution yielded from this review was remarkably small, peaking at around 1.3 decades. In response to this low display of adherence to fractal laws, they pose the questions "Do all power laws that are limited in range represent fractals? Is it justified to term them as such?"[13] The answer comes in the simplification of a complex system to a more concrete understanding. Utilizing a power law to condense complex systems can make the analysis of the properties and correlation of information relating to the systems much simpler, however, data's congruency with this sort of a power law, particularly on such a limited scale, does not necessarily constitute what we would define as 'fractality.'

The purpose of detailing this account and criticism of the application of the term 'fractal' in experimental reports is not to deride its usage. Rather, it is to highlight the discrepancy between what Mandelbrot defined as the nature of a fractal set and where it is useful to apply fractal vocabulary, perhaps even in the face of the perfect mathematical definition. The paper does concede many legitimate reasons for the continued application of the term. The simplification of complex geometrical structures so that they may be usefully studied and correlated, the application of a fitting language to describe forms, and the qualitative self-similarity observable in even limited-range irregular objects are cited. The culminating point of utility is somewhat fatalist, though perhaps a useful progression for our purposes – we are already so far gone in our rampant fractal categorization that the word has shifted from its original definition.

The implications of expanding the application of a term can be positive or negative. On the one hand, expanding it reduces the specificity with which we can refer to a form or a system. On the other, wider application can allow for a better

[13] Anir, David, *et al*. IBID. 39.

understanding of a larger variety of systems. As recognized in the paper, “A drift from an original meaning of a concept is common in science, representing the adaptability of the original ideal definition to realistic restrictions that emerge when put to practice.”[14] In the end, the paper takes a negative stance on the expansion. However, here is where I diverge.

There must be a distinction between a fractal set as put forth by Mandelbrot and natural fractality, as it seems that limited-range self-similarity is inherent to natural fractal forms. There are physical cutoffs in nature that limit the possibility of a fractal structure, the lowest being whatever the most basic building block is (atoms, molecules, etc.) and the highest cutoff being limited by any combination of limitations of mechanical strength, anisotropic growth fields, resource depletion, etc. Therefore, except under impossibly ideal circumstances, a fractal set will not grow in nature. The criticism of the application of a term to natural forms for which there is no perfect example is a fallacy. The question asked should not be if the form is truly fractal, but if it is useful to describe and analyze it in such a way, which the critical paper clearly concedes. So, is the geometry of nature truly fractal? “Maybe no, but we will keep saying it is.”[15]

The issue of the application of the term out of the way, we can delve into the qualities of these natural fractal forms. Early in *The Fractal Geometry of Nature*, Mandelbrot quotes extensively from Jean Perrin (who later won a Nobel Prize for his work on Brownian motion) in a 1906 philosophical manifesto. It is a little known work, outside of being quoted in Mandelbrot’s essay. To summarize, Perrin discusses physical entities on which a tangent to any particular point seems impossible to draw because if

[14] Anir, David, *et al.* IBID. 40.
[15] Campbell, Paul J. "Is the Geometry of Nature Fractal?" *Mathematics Magazine* 71.3 (1998): 233.

the infinitesimal detail involved, such as a coastline, Brownian motion, etc. The idea put forth is that “we find [these entities] growing more and more irregular as we increase the magnification of our necessarily imperfect image of the universe.” He goes on to hypothesize, “One might encounter instances where using a function without a derivative would be simpler than using one that can be differentiated. When this happens, the mathematical study of irregular continua will prove its practical value...However, this hope is nothing but a day dream, as yet.”[16] Mandelbrot chastises the fear of irregularity and timidity towards chaotic forms. “Mathematicians are to be praised for having devised the first of these sets long ago, and scolded for having discouraged us from using them.”[17] The observation of and even beginnings of mathematical understanding of these complex forms existed before Mandelbrot because they are integral to the understanding of our world.

The purpose of this essay is not to prove that the geometry of nature is, in fact, fractal. Mandelbrot’s rigorous work and a veritable surfeit of supporting literature since have adequately done the job without my assistance. Again, our aim is to examine the application of the term to forms. Alongside Mandelbrot’s examination of natural fractal forms – such as a coastline, a snowflake, a galaxy, or a tree – he pairs them with correlating fractal sets. This comparison makes clear to us the sort of qualitative self-similarity exhibited in the world, rather than the mathematical self-similarity requisite for a fractal set.

For example, when looking at a snowflake or a coastline, he examines a Koch curve. The Koch curve is one of the earliest fractal formulations to have been created,

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[16] Mandelbrot, Benoit. *The Fractal Geometry of Nature*. 9.
[17] Mandelbrot, Benoit. *The Fractal Geometry of Nature*. 9.

and is defined as a continuous curve without tangents. It is formed with building blocks of equilateral triangles, each successive generation being one-third the size of the previous generation, positioned in the middle of each side of the previous generation. When truly infinitely recursive, this curve has infinite perimeter. We can see the progression of generations in the MATLAB rendered images of 1, 2, 3, and 4 generations of the Koch curve in Figure 1 below.

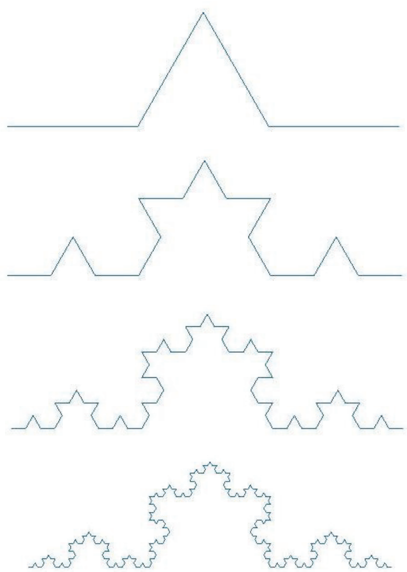

Figure 1: A Koch curve developing through four generations.

As we can see, the fractal continues on itself in such a way that at any conceivable scaling in the infinite rendition of this form, we could see the image of the first generation. This is analogous to the infinite detail found in the curve of a coastline or the recursive complexity of a snowflake. Indeed, were we to create a closed figure out of the fourth generation, it would roughly fit what we see in the natural world. This is one instance of a fractal set being an idealized form of a natural form. We see such similarity in trees, with a fractal tree being built in much the same manner as the Koch curve, though with stems and branches, rather than triangular disruptions in a line segment. These conceptual renditions are, unlike nature, flawless.

We have determined that despite the limited scaling range of fractals observed in nature, the application of the term is useful for the correlation of data and forms. The qualitative self-similarity is enough that the term "fractal" is usefully utilized. The fractal sets that can be applied to natural forms clearly bear a qualitative similarity in such a way as to serve as models for generational growth according to the realization of a probability distribution, resulting in stochastic growth of the system. With this understanding of the useful application of fractal sets to natural forms, we can distinguish between kinds of fractals in materials and art.

In Materials

An understanding of and ability to manipulate fractal forms is useful for the tuning of a variety of physical properties in materials that exhibit such structures. Fractal materials, much in the same way as the Koch curve's approaching infinite perimeter, have an incredibly high surface area. Surface morphology influences any of the properties one might want to exploit out of a material, be they electrical, magnetic, optical, chemical, or mechanical. As a case example of the fractal analysis of materials and how an understanding can be useful for the implementation of materials, we shall examine work done on niobium thin films for the purposes of shielding of superconducting radio frequency (SRF) cavities in linear accelerators.[18]

To analyze the fractal structure of the thin films, data from AFM scans (Figure 2) representing the thin film surface on the x-y plane and a z-value for surface depth variations was scanned. These images were analyzed line by line, applying a continuous wavelet transform. This essentially consists of applying a wavelet, which acts as a filter to focus on certain features and patterns, to a line scan from the AFM image. Coefficients that apply to a basis function, which vary according to different scaling parameters were found. This sort of analysis is well suited to such a situation, particularly in comparison to harmonic analysis, because patterns need not be periodic or follow some defined frequency, but may be found by a localized wave packet's scaling.

For this particular analysis, a "Mexican Hat Wavelet" was utilized, though others may have served as well or better. Other wavelet shapes may better resemble the signal from the line scan and would then give different coefficients for the scaling and

[18] A slightly more detailed description of methods and results can be found in Appendix A.

translation of the wave packet, yielding a different representation of the scaling dynamics. The scope of the project limited the analysis to one form, however. Compelling results were nonetheless attainable with the wavelet utilized.

The thin films exhibit scaling of the surface features, apparent when one qualitatively compares AFM scans of films of different thicknesses. There are uniform patterns in the surfaces of these thin films, which increase in dimension as the film increases in thickness. This suggests that there ought to be some kind of universal rule of scaling for thin films in this growth pattern that may apply to the observed features. The self-similar nature of these features suggests a fractal-like scaling, wherein base structures join together as a film grows to form similar structures of larger scale.

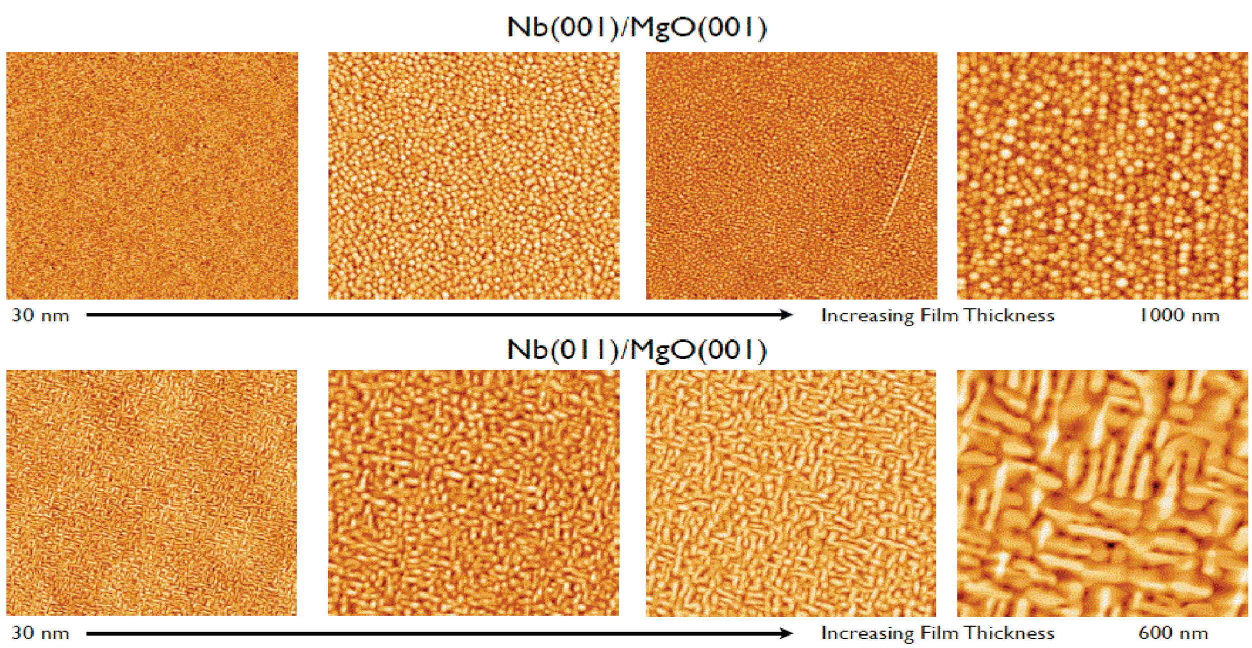

Figure 2: Features of surface morphology have a positive correlation with the film thickness, suggesting fractal growth.
D.B. Berringer, et al. *Wavelet Analysis of Surface Morphology in Epitaxial Niobium Thin Films.* 2014

An understanding of the process by which the data was analyzed is useful in understanding the results. The actual data crunching for the experiment was simple, but time consuming. Each AFM scan was 1024x1024 data points (over one million, total).

Scanning and analyzing a full image would have taken an immensely long time. Instead, because the pattern is relatively uniform across the surface of the thin film, we found that the data fit itself to the same coefficients very quickly with only few line scans. To be safe, after determining how much of a difference scanning more lines made to the precision of the measurement, 256 lines were scanned.

The program worked in multiple steps. It would initially scan one line of the data from the AFM image, plotting the height as a function of x. The wavelet transform is then applied to the line scan, which yields a wavelet scalegram, telling the distribution of variances of the coefficients. This wavelet scalegram, after averaging and calculating the structure factor, yields an averaged scalegram. This is what tells the most about the targeted scaling features of interest to determine the fractal nature of the thin film growth. For the self-similar properties that we observe, we expect to see linear regions in the averaged scalegram, the slope of which is proportional to the scaling factor. We observed that the scaling factor changed at a certain point, such that the averaged scalegram had two different slopes depending on the wavelet scale.

The conclusion drawn is that there are two static scaling parameters, the behavior changing at a particular scale. The evolution of the surface in the synthesis process takes on a scale invariant pattern in forms of the microstructure, resulting in these surface patterns. The determining of scaling parameters can give insight into what causes certain features to arise, allowing for alterations in the synthesis process to tune the structure to its needs.

This study of the fractal growth of thin films is simply one example as to how man attempts to understand and manipulate fractal patterns to serve an end. These man-made materials still conform, however, to natural laws despite our manipulation. All matter is subject to myriad, inescapable forces. We may combat the randomness of nature through understanding of fractal mechanisms, but nature takes over and dictates a certain stochasticism. The best efforts simply tighten the probability distribution. Materials, therefore, exist under the umbrella of natural fractals, roughly fitting the fractal set. We strive towards the fractal set in our understanding and manipulation of materials, but only get so far. Man-made fractals are not limited to our manipulation of matter for materials, however. Art, conceptualized and created by the human mind, forms orders and constructions of a fractal nature that can be analyzed in much the same way.

In Art

"Non-linear dynamics is now become a multidisciplinary domain of research."[19] So begins Bartosiak's treatment of Tom Stoppard's *Arcadia*, which we will get to a little further along. Fractals are evident in much of art, primarily from the 1980's onwards, particularly in western art. The introduction of the fractal form to art makes sense on this timeline, coming into vogue parallel with the reclamation of chaos occurring in the sciences. We do see some self-similar, fractal-esque patterns dating even further back, however. We see it all over as humanity attempted to express the chaos of the forms around them. From ancient Hindu temples, to the interior of the main dome of the Selimiye Mosque in Turkey, which dates back to 1574, to Australian aboriginal painting dating back tens of thousands of years, the suggestion of fractality in art is omnipresent. This inclusion of fractality in art, intentional or unintentional, suggests an inherent understanding of the complex structures that comprise our world. The ability to apply the dialect of fractal geometry to art widens the reach of this tool we have found to correlate chaos.

Fractals have been present in art since eons before the geometric language through which we discuss it was formalized. This only makes sense, as art has a tendency to meditate on and reflect the natural world, wherein we find these fractal forms. Examples of fractal structures in art abound, one of

Figure 3: Hokusai, *The Great Wave of Kanagawa*
www.wikipedia.org

[19] Bartosiak, Mariusz. "Towards a Fractal View on Drama: The case of "Arcadia" by Tom Stoppard." Degres-Revue de Synthese a Orientation Semiologique. 102-03 (2000): f1-f13

the most famous being Katsushika Hokusai's *The Great Wave of Kanagawa* (Figure 3) from the early 1830's. The repetitive form of miniature waves in the crest of the larger wave is immensely suggestive of a fractal tree. Repetitive dots and symbols building upon each other to make larger forms contributing to a whole work purvey ancient cultures' art all over the world.

Reflections of cultures' world-views are present in the fractality of certain constructs they build for themselves. One particularly interesting example is ancient Hindu temples. They would very often build a tower surrounded by smaller towers, surrounded by still smaller towers, similar in structure on a multitude of scales. An example can be seen in Figure 4. The self-similar construction of the tower is suggestive of the Hindu religious system. The smaller constructions rising and joining the larger, singular, central entity is analogous to their belief of reincarnation, rising through levels of consciousness to join the all-encompassing transcendence above. The fractal nature of their temples reflects the fractal nature of their belief system. Before the language and concepts were even available to them, humans were expressing their perception of the world through fractal geometry. It is clear there is something intuitive and understood about the way in which we perceive and express complex forms.

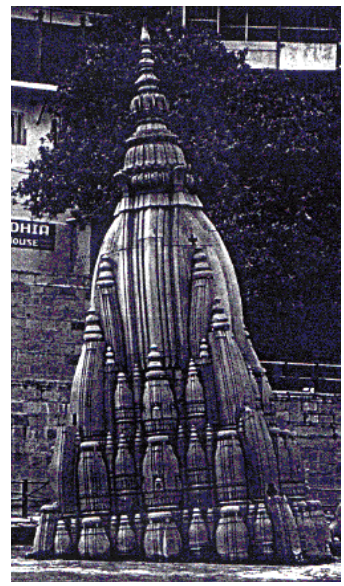

Figure 4: Ancient Hindu temple. http://classes.yale.edu/fractals/panorama/Architecture/IndianArch/IndianArch.html

Painter Jackson Pollock is famous for his unorthodox drip painting technique that has been recognized as a pivotal movement in the progression of modern art. The seemingly random dripping of colors on canvas is actually very precisely organized and structured. A 1999 article in *Nature*[20] detailed a fractal analysis of his work, such as *Alchemy* (Figure 5). The researchers found that there was a distinct fractal arrangement in the figures in Pollock's work, reflecting the complexity and form of nature.

Figure 5: *Alchemy,* Jackson Pollock (1947)
www.jackson-pollock.org

To analyze the fractal nature of his paintings, they used a 'box-counting' method. Essentially this amounts to making a grid of increasingly smaller squares and counting the number of boxes containing the painted pattern. The log-log plot of number of squares with the painted pattern versus the size of the squares yield the fractal dimension *D* (the index describing how the complexity of detail in a fractal pattern varies with scale). Interestingly, they found that there were two distinct values of *D* over a small scale and over a larger scale. The suggested reason behind these two scales comes from an analysis of video of him painting. The small scale fractal dimension arises from the actual dripping process of the paint while the larger dimension is determined by Pollock's movements around the large canvas.

The detailed, mechanical understanding of the fractal patterns exhibited in Pollock's painting may seem like an overly technical analysis of a splatter painting, but the ability to correlate data over time gives insight into the progression of Pollock's method over time. The fractal dimension *D* increases over time, gives objective evidence of a development of his drip technique over time. This progression of fractal qualities in his paintings shows increasing complexity, systematically expanding the chaotic forms.

[20] Taylor, Richard, *et al.* "Fractal Analysis of Pollock's Drip Paintings." *Nature*. June 3, 1999. 399.

This objective study of the progression of his techniques gives us a consistent theme by which to analyze his work over time and is evidence of the structured, deliberate reflection of chaotic natural forms in his work.

Figure 6: *Flashes*, Francois Miglio
www.miglioart.com

In addition to the analysis of modern works that exhibit a fractal pattern, the development of fractal geometry gave rise to a new form of art altogether, deliberately using fractal forms in the works. Much of fractal art is actually created through computer renderings, but some is produced in a more traditional fashion. French artist Francois Miglio is a notable figure in fractal art, producing works of what he calls "fractal syncretism." His method is discussed on his website: "He creates several layers with oil pastels, and with a knife he repeatedly and meticulously carves some extremely precise patterns."[21] This layered structure and the combination of smaller forms to build much larger forms reflects the conventional fractal structure. As can be seen in *Flashes* (Figure 6), the image is composed of many tiny dots, reminiscent of impressionism. These dots comprise lines and curves that in turn comprise larger forms that combine to create a full image. As he says in his manifesto, "This art integrates the notions of fractal, rather than Euclidean, geometry and is engaged by the creative and chaotic process of dreaming."[22] Through his art he works to express the complexity of the world, attempting to reflect large and small natural forms as one.

[21] Miglio, Francois. "The Art of Francios Miglio." *MiglioArt*. Web.
[22] Miglio, Francios. IBID.

Thus we see that for visual art, fractals appear both as an inspiration and integral artistic component of works and as a useful tool by which we can analyze and correlate complex, chaotic works. This is as true for ancient architecture and paintings as it is for modern works. Art and science converge at this point, exploring chaotic structures present in the natural world.

Fractals do not manifest simply in visual art like painting and architecture. Dance, music, and drama all have instances of fractal structure. One can see the fractal of individuals building into units building into groups in the form of the whirling Dervish dances. Philip Glass is notorious for his repetition of forms, building slowly upon each other in a way that could be described as fractal. Of particular interest are instances of fractality in drama, as it is significantly less of a deliberate structural form in which such geometry would manifest.

Drama is, in form, inherently fractal. Each play, each instance within the surfeit of dramatic literature is a fragment of human culture. The play itself is divided into fragments of moments, into acts and scenes. These scenes are comprised of events, moments of conflict between individual entities that in turn have their own individual motivations. Bartosiak puts this concept forward in his earlier referenced paper, culminating in an examination of a peculiarly interesting play in both the realms of humanities and mathematics, *Arcadia* by Tom Stoppard. This play oscillates back and forth between the nineteenth and twentieth centuries, alternating scenes within the same space. Throughout the play, items from each period anachronistically remain on the stage. They are not addressed by the characters, but contribute to the whole construct. This iterative building of fragmented contributions to a whole has a fractal essence to it.

The standard constructions of time, space, and character are divided into a binary system of the two worlds portrayed. Eventually, the oscillation ceases and all characters exist simultaneously in the space, chaotically collapsing the temporal division.

The fractal construction is reflexive of the content of the play. *Arcadia* follows two mathematicians in these time periods, struggling with the concepts of chaos and the patterns of the world. They study the world around them and observe the chaotic fragmentation of forms all contributing to the whole. Eventually, the actions and observations of characters in the earlier time period have butterfly effects on the later characters, events compounding upon each other in a chaotic, fractal way to their end result.

A similar instance of fractality in art through temporal manipulation is exhibited in my concurrent project of the analysis and direction of Michael Frayn's play *Copenhagen*, on which another paper is being written. Detailed analysis of Frayn's presentation of perspective and the theatrical theory behind the staging can be found there, however, a brief discussion will do us good in understanding how a fractal perspective can help us understand complex presentations of forms even in art. We will first look at the content of the show that necessitates the form.

Copenhagen is about one fateful evening in September, 1941, when Werner Heisenberg travelled from Germany to Nazi-occupied Denmark to visit his former mentor, Niels Bohr. Heisenberg was the leader of atomic research in Germany, having already contributed his uncertainty principle to the Copenhagen Interpretation of quantum physics. The danger of travelling outside of Germany to visit Bohr, the proximity of this visit to the invention of the atomic bomb that ended the war and forever changed international dynamics, and the physicists' own ambiguity about the events of the evening has baffled journalists, historians and scientists ever since. Frayn writes the play from a space in which all three characters are already dead, examining and reinhabiting memories of the evening to understand what the purpose of Heisenberg's visit was.

The play utilizes an unconventional staging for this conversation. For the purposes of storytelling, it seamlessly transitions back and forth between 1941 and this abstract "afterlife" space with characters both commenting on moments and observations within their memories in 1941 from the perspective of their "afterlife" selves and occasionally directly addressing the audience, reporting observations and analyses thereof. Frayn's surfeit of literary work exhibits a consistent theme of the manipulation of perspective. This manipulation manifested in our presentation of *Copenhagen* as a "postdramatic" staging (discussed in more detail in my other paper). This essentially amounts to the segregation of setting from concrete place, of presentation of time from chronology and linearity, and character from context. We developed a staging in which the action of the play is analogous to the presentation of an experiment, the results of which are reported to the audience as observer. The intent was to create an audience perspective from which they could perceive simultaneously and equally each character's individual perspective. This breaks from the traditional dramatic presentation where the audience is meant to follow the arc, the narrative, and the conflicts of one or a small set of characters, but from singular perspectives at a time. This sort of omnipotent view is well put in Frayn's description of the perspective of God:

> "God saw it all. Saw it all in one go, continuously and eternally. And since God was everywhere, he saw it not in perspective, not from some

> particular viewpoint, but from every possible viewpoint. From all sides of a cube simultaneously, for example. From an angle of ninety degrees to each of those sides – from an angle of one degree, eighty-nine degrees, seventeen degrees. From a millimetre off and a mile off. From every point inside the cube looking out."[23]

The significance of this audience perspective in fractal terms is apparent. The audience has a singular perspective upon the three individual presentations of the characters' perspectives on stage. These characters, in their turn, have their own perspectives, inherently limited by their station in the universe. The fallibility of memory and the limitation of human observation are consistent themes throughout the play. Margrethe, Bohr's wife, says of self-knowledge, of observation and understanding of one's own position, "If it's Heisenberg at the centre of the universe, then the one bit of the universe that he can't see is Heisenberg."[24] These characters discuss their observations of and interactions with multitudinous other entities that never enter the stage, all of which have their own perspectives and make their own observations from their own inherently limited station at the center of the universe. In this presentation, a fractal tree is formed. From the trunk of the audience's single line through to the simultaneous presentation of three characters' perspectives, *to* each one of those individual characters' perspectives on the world and other individuals around them, who in turn have their own limited perspectives.

This sort of recursive observation of observations, perspective on perspectives, makes for an increasingly limited understanding of objective truth as the generations

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[23] Frayn, Michael. *The Human Touch*. 32.
[24] Frayn, Michael. *Copenhagen*. (New York: Anchor Books, 2000). 72.

propagate. Limitation compounds upon limitation, resulting in an ultimate indeterminacy. This parallels the content of the show, which discusses both Bohr's theory of complementarity (an inherent link between physical properties which means that the more accurately one is observed, the less accurately another can be) and Heisenberg's uncertainty principle (intimately linked to complementarity, asserting a fundamental limit to the degree of accuracy complementary properties can be measured simultaneously.) We see this inherent indeterminacy in the subdivision of matter into increasingly smaller units (which, conveniently, is another fractal process), bulk materials into the microstructures that comprise them, structures into molecules, molecules into atoms into subatomic particles into quarks and other such entities that boil down to nothing more than probability. The solidity of anything is only a function of the scale on which we perceive it.

Mandelbrot himself addressed this relationship between the observer and the number of dimensions observable dependent upon their perspective. "The notion that a numerical result should depend on the relation of object to observer," he said, "is in the spirit of physics in this century and is even an exemplary illustration of it."[25] Every subsequent scale on which an object exhibits its fractal form "pursues the ever-receding horizon of exactitude…by a series of closer and closer approximations."[26] This fractality of perception is utilized to demonstrate the indeterminacy inherent to the human position in the universe. It follows a natural form, analogous to the subdivision of matter into discrete building blocks, ultimately resulting in a fundamentally fuzzy universe. Because

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[25] Mandelbrot, Benoit. *The Fractal Geometry of Nature*. 18.
[26] Frayn, Michael. *The Human Touch*. 105.

of this infinite fractal subdivision, all things boil down to, as Heisenberg says in the closing lines of the play, "that final core of uncertainty at the heart of things."[27]

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[27] Frayn, Michael. *Copenhagen*. 94.

Conclusions

The debate about the nature and applications of fractal geometry is as chaotic as its subject. The formal definition of a "fractal" is under contention, but it is clear through its immensely varied applications and its omnipresence in the natural world, materials, and art that the rigorously defined fractal set must be able to be liberally applied to much less structured forms. The universe exists in the language of mathematics in such a way that even the most complex and chaotic forms can be correlated with the proper dialect.

The young mathematician Thomasina in *Arcadia* says "If you could stop every atom in its position and direction, and if your mind could comprehend all the actions thus suspended, then if you were really, *really* good at algebra you could write the formula for all the future…"[28] The universe, when subdivided to the smallest possible unit of fractal existence, we see the fuzzy inconstancy that Frayn describes.

> "We look closer, and see that each particle is in itself a world in flux, a hierarchy of still smaller particles – of particles that are not precisely particles, but additionally and alternatively wave formations, fluctuations in probability, whose precise state can never be fully expressed…These gritty grains of sand, so eminently and geometrically and tangibly *there*, are analysable into constituents whose defining characteristics can never be completely and precisely determined."[29]

When the entire universe exists in such a state of rough indeterminacy as to be nothing but probability, what business do we have so rigorously defining such a useful lens as to limit its application? The perspective of fractal geometry is paramount to the

[28] Stoppard, Tom. *Arcadia*. New York: Faber and Faber, 1993. (10)
[29] Frayn, Michael. *The Human Touch*. (12)

simplifying, analyzing, and understanding of myriad complex systems. A formal, homogenous classification of fractal geometry is in opposition with the inherent nature of the irregular, fragmented systems it describes.

Ultimately, we are left with a question of utility. The implications of fractal geometry being so pervasive in all manner of physical, mathematical, conceptual, and artistic systems is indicative of the inherent chaos of the universe. There exists a lens by which we can consolidate and classify, correlate and understand. We just have to be willing to accept the rough, fuzzy, ultimately indeterminate nature of our universe, and apply our constructs in ways that are useful if somewhat fluid. As Mandelbrot said, "Beautiful, damn hard, increasingly useful. That's fractals."[30]

[30] Mandelbrot, Benoit. "24/7 Lecture on Fractals." *2006 Ig Nobel Awards*. Improbable Research.

Appendix A: Methods and Results of AFM Wavelet Analysis

This work was done by Douglas Berringer, et al., a summary of which can be found here: D.B. Berringer, et al. *Wavelet Analysis of Surface Morphology in Epitaxial Niobium Thin Films.* 2014.

The wavelet analysis done was to examine surface morphology of the thin films as a function of thickness with two different epitaxial growth patterns (Nb100/Nb110) on MgO001. The goal was to determine a fractal-like scaling pattern through wavelet analysis of AFM images.

We had data from AFM scans representing the thin film surface on the x-y plane and a z value for surface depth variations. We analyzed these images line by line, applying a continuous wavelet transform. This essentially consists of applying a wavelet, which acts as a filter to focus on certain features and patterns, to a line scan from the AFM image. We find coefficients that apply to a basis function, which vary according to different scaling parameters. This sort of analysis is well suited to such a situation, particularly in comparison to harmonic analysis, because patterns need not be periodic or follow some defined frequency, but may be found by a localized wave packet's scaling.

We utilized a "Mexican Hat Wavelet," though others may have served as well or better. This is one area in which we may be able to take this wavelet analysis further. Other wavelet shapes may better resemble the signal from the line scan and would then give different coefficients for the scaling and translation of the wave packet, yielding a different representation of the scaling dynamics.

The thin films exhibit scaling of the surface features, apparent when one compares AFM scans of films of different thicknesses. There are self-similar patterns in the surfaces of these thin films, which increase in dimension as the film increases in thickness. This suggests that there may be some kind of universal rule of scaling for thin films in this growth pattern that may apply to the observed features. The self-similar nature of these features suggests a fractal-like scaling, wherein base structures join together as a film grows to form similar structures of larger scale.

The actual data crunching for the experiment was simple, but time consuming. Each AFM scan was 1024x1024 data points (over one million, total). Scanning and analyzing a full image would have taken an immensely long time. Instead, because the pattern is self-similar, we found that the data fit itself to the same coefficients very quickly with only few line scans. To be safe, after determining how much of a difference scanning more lines made to the precision of the measurement, we decided on scanning 256 lines.

The program worked in multiple steps. It would initially scan one line of the data from the AFM image, plotting the height as a function of x. We could only do one dimensional line scans with this particular program, but the extension of the concept to two dimensional scans is another direction to take this project in the future. The wavelet transform is then applied to the line scan, which yields a wavelet scalegram, which tells the distribution of variances of the coefficients. This wavelet scalegram, averaged and after calculating the structure factor, yields an averaged scalegram. This is what tells the most about the scaling features that we want to analyze. For the self-similar properties that we observe, we expect to see linear regions in the averaged scalegram, the slope of which is proportional to the scaling factor. We observed that the scaling factor changed at a certain point, such that the averaged scalegram had two different slopes depending on the wavelet scale.

In conclusion, we determined that the sample set of line scans need not be extended to the whole AFM image, but can be isolated to as little as a quarter of the data points and yield accurate results, reaffirming the fractal self-similarity of the surface. A linear fit for self-affine features is expected on the log-log plot of the averaged scalegram referenced above, and we saw two distinct slopes with a change in behavior part way through. We see two distinct scaling parameters, meaning the static scaling behavior changes at a particular wavelet scale. This differed between the surfaces, but the important feature is that there was a marked change.

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