Sums of Convergent Series

I. Introduction

Often the functions used in the sciences are defined as infinite series. Determining the convergence or divergence of a series becomes important and it is helpful if the sum of a convergent series can be determined. Most of the tests taught in Chapter 11 can determine convergence but do not allow us to find the sum of the series. One exception is a convergent geometric series which has the sum,

$$s = \frac{a}{1-r} \tag{1}$$

where a is the first term of the series and r is the common ratio.

For any series, if there is a formula for the n^{th} partial sum, s_n, the exact sum is

$$s = \lim_{n\to\infty} s_n \tag{2}$$

if the limit exists. If a series is a telescoping series, we can find a formula for s_n and use (2) to determine its convergence and sum.

Series that converge may or may not have a known sum, however. For many series, we can only estimate the sum.

II. Estimating Sums

Given a convergent series, any n^{th} partial sum may be used to approximate the sum of the series. Some helpful calculator steps are at the end of this lab.

Example 1: We know the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. How would you support this conclusion?

$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+...$, is convergent and one possible approximation of the sum is $s_4=1.4236\overline{1}$. Of course, a better approximation is $s_6=1.4913\overline{8}$ and $s_{100}=1.6349839...$, is better still. In fact, we know the n^{th} partial sum becomes a better approximation as n increases. Why?

III. Error

Every approximation is just that…an approximation rather than the exact value. So every approximation has an associated *error*. In the previous example, the first four terms of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are used to calculate s_4, this approximation has an error equal to $a_5+a_6+a_7+...$. Generally, the error in using s_n to approximate the sum of a convergent series is equal to the sum of all the remaining terms. That is,

Error = actual sum – approximate sum

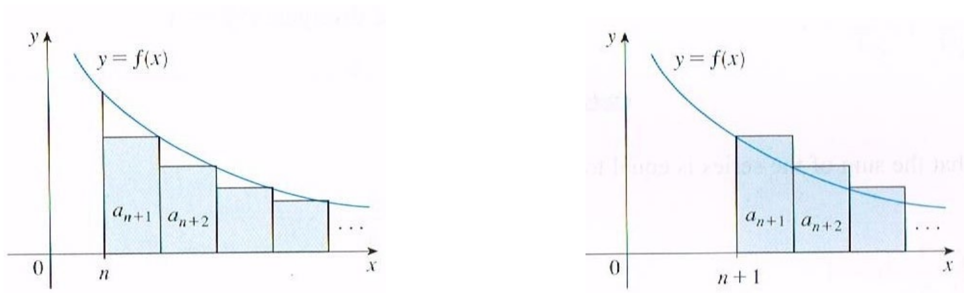
or

Error = s - s_n

This error is called the *remainder* which makes sense as it is made up of the remaining terms not used to calculate the approximation, s_n. The notation R_n denotes the error in using s_n to approximate s. So,

$$R_n=s-s_n=a_{n+1}+a_{n+2}+a_{n+3}+... \quad (3)$$

If we cannot find the exact sum of a series, then we certainly cannot find the exact remainder associated with an approximation. However, if the remainder can be bound, we can improve the approximation of the sum. If the Integral Test can be used to determine convergence of a series, it can also be used to find bounds for R_n. The sketches below illustrate how R_n is bound between two area calculations.



So the error is bound as follows:

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx \qquad (4)$$

The exact sum is somewhere between the approximation plus the smallest error and the approximation plus the largest error. That is,

$$s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx \qquad (5)$$

If an Alternating Series is convergent, the error in approximating the sum with s_n can also be bound. The error is no greater than the absolute value of the first "neglected term" in s_n.

Example 2: The alternating harmonic series shown below is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$$

If $s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$ is used to approximate the sum of the series, we know the remainder is equal to the sum of all the remaining terms. Once again, we cannot find the exact remainder but we *can* find an upper bound for the remainder. Notice what happens if we approximate the remainder, adding one more term in each approximation:

$$R_5 \approx -\frac{1}{6}$$

$$R_5 \approx -\frac{1}{6} + \frac{1}{7} = -\frac{1}{42}$$

$$R_5 \approx -\frac{1}{6} + \frac{1}{7} - \frac{1}{8} = -\frac{25}{168}$$

We see $\left|-\frac{1}{6}\right| > \left|-\frac{1}{42}\right| > \left|-\frac{25}{168}\right|$. The error continues to get smaller and smaller, as more terms are added. Thus, the error (in absolute value) will be no greater than $b_6 = \frac{1}{6}$.

Generally, for series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ that satisfy the conditions of the Alternating Series Test, the size of the error is

$$|R_n| = |s - s_n| \le b_{n+1} \qquad (6)$$

IV. Helpful Calculator Stuff

We can list values of the sequence of terms and the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$.

To clear y= and lists: **2nd** , **MEM**, **4**, **enter**

Bring up table as shown: **STAT**, **ENTER**

Arrow over and/or up to highlight L_1 : **2nd**, **LIST**, **OPS**, **seq(x,x,1,100,1) ENTER**

This displays the *n*-values from 1 to 100 in L_1.

Arrow right and up to highlight L_2. Enter the a_n formula, replacing *n* with L_1, then **ENTER**.

This displays the sequence of terms in the series.

Arrow right and up to highlight L_3. Enter: **2nd**, **LIST**, **OPS**, **6**, **ENTER**, **2nd**, L_2, **ENTER**

This displays the sequence of partial sums of the series.

L_1 n	L_2 terms	L_3 partial sums
…	…	…

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You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. **No calculators unless otherwise noted.**

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

(a) Use partial fraction decomposition to express a_n as a combination of two fractional expressions.

(b) State what type of series this is.

(c) Expand the series to find a formula for s_n .

(d) Find the sum of the series, if it exists. (Hint: this involves finding a limit.)

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2. Consider the series, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^{k-1}}{3^{2k+1}}$. (a) Expand the series to 5 terms.

(b) State what type of series this is.

(c) Determine if the series converges or diverges. Your work should include steps to justify your conclusion.

(d) If the series converges, find the exact sum if this is possible.

3. (a) Show the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$ converges by using the Alternating Series Test.

(b) Approximate the sum of the series using s_6. (calculator)

(c) Find the largest error expected in using this approximation for the sum. (fraction or decimal)

4.(a) Show the series $\sum_{n=1}^{\infty} ne^{-2n^2}$ satisfies the required conditions for the Integral Test.

(b) Use the Integral Test to show the series converges.

(c) Find the largest error expected in using the 4^{th} partial sum to approximate the sum of this series. (Final answer should be an exact value, no calculators.)