Introduction to Sequences and Series

I. Sequences

A sequence is a list of numbers in a definite order. While finite lists are vitally important in everyday life, in calculus we concentrate on infinite lists. This prospect causes an immediate problem. In principle, finite lists can always be written down for inspection. There is not enough time or paper to write down an infinite list so we have to resort to some trickery. One standard method (there are others) depends on the characteristic of ordering. One may say, for instance, that among the entries in the list, there is a first, a second, a third and so on. The idea, then, is not to try to write down all the entries, but rather to provide a formula for *computing* any particular entry. A formula might look like this: $a_n = n^2 + 2$

Notice the formula for the entries in a sequence is nothing more than a function whose arguments are restricted to integers, usually the positive integers beginning with 1.

Someone who wants the 10th entry in this list just computes: $10^2 + 2 = 102$

Someone else who wants the 3874th entry in the list computes: $(3874)^2 + 2 = 15{,}007{,}878$

The notation a_n describes a "generic" entry in the list, the *n*th entry, where *n* is any positive integer. Given such a formula, a user may write down as many (but only finitely many) of the entries as are desired. Sometimes a more casual version of the formula method is used. For example, the "even integers beginning with 14" would describe the sequence $\{14,16,18,\ldots\}$. This partial list, by itself, describes to most people what sequence is under discussion. While the structure of that sequence may be clear to everyone, a textbook problem is liable to give one or the other of the previous descriptions and then demand that the student construct a formula for subsequent manipulation. In this case, for example, a textbook problem would be asking for the formula $a_n = 2n + 12$ for this sequence.

(WORK EXERCISE 1)

II. Convergent Sequence?

Why the interest in infinite sequences? One reason is the question of approximation. Consider the sequence $S = \{1, 1.4, 1.41, 1.414, \ldots, s_{n,\ldots}\}$, a casual method of describing the sequence of successive decimal approximations to $\sqrt{2}$. Because the entries, s_n, in this sequence are getting closer and closer to a target, $\sqrt{2}$, we say the sequence converges to $\sqrt{2}$. A more precise description of the notion of convergence says we can make $\sqrt{2} - s_n$ as small (but not zero) as we please by going out far enough in the sequence.

Notice it is the long-term behavior of the sequence that determines its convergence. For instance, the sequence $\{11, 76, 3, 1, 1.4, 1.41, 1.414, ..., s_{n,...}\}$ still converges to $\sqrt{2}$, even though the front end of the sequence is messed up. There are "trick" methods to verify that a given sequence converges *without identifying the target*. But, we will proceed with the method of actually working out the target, a value that is called the *limit* of the sequence.

(WORK EXERCISE 2)

III. Series

Now, we want to use one sequence to generate another, and not just for whim. Look at the sequence

$A = \{1, .4, .01, .004, ...a_n, ...\} = \left\{1, \frac{4}{10}, \frac{1}{100}, \frac{4}{1000}, ..., \frac{d_n}{10^n}, ...\right\}$. What are we describing so casually?

The answer might be clear; these are the decimal "additive corrections" that, beginning with 1, produce our earlier sequence $S = \{1, 1.4, 1.41, 1.414, ..., s_{n,...}\}$, the decimal approximation to $\sqrt{2}$. The n^{th} decimal digit in this expansion is d_n. In other words,

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + .4 = 1 + \frac{4}{10} = 1.4 \\
s_3 &= 1 + .4 + .01 = 1 + \frac{4}{10} + \frac{1}{100} = 1.41 \\
s_4 &= 1 + .4 + .01 + .004 = 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} = 1.414 \\
&\vdots \\
s_n &= 1 + \sum_{i=1}^{n} \frac{d_i}{10} \\
&\vdots
\end{aligned}
$$

Thus, we use the entries in one sequence to produce, by successive additions, the entries in a second sequence. It is not a bad idea to think of these successive additions as a sequence of "correction" to the sum, just as in the case of the decimal expansion for $\sqrt{2}$. There is a standard notation for this sequence of corrections. If the first sequence is $A = \{a_1, .a_2, ..., a_n, ...\}$, then the second sequence could be written $S = \left\{a_1, a_1 + a_2, a_1 + a_2 + a_3, ..., \sum_1^n a_i, ...\right\}$ or even just $\sum_1^\infty a_i$, the common series notation.

The new sequence of partial sums is generated by the *infinite series* which is the sum of the terms in sequence A. In certain cases, it is possible, with some labor, to find a formula for $s_n = \sum_1^n a_i$. We hope such a formula allows us to work with the series. Usually, it is too difficult to find formulas like this and we instead, employ another method to determine if a series has a sum. One thing is pretty clear…if the terms, a_n, in the original sequence are to be used as additive corrections, they must be getting smaller quickly. Otherwise, the successive additions will make the sum jump around too much!

(WORK EXERCISE 3)

IV. Sequences and Series: Don't confuse them!

Although divergent series and sequences find use in more advanced work, our main focus lies in those that converge. Many students are confused by the variety of methods used to verify convergence.

There is a single fundamental principal that controls all approaches to convergence, whether we are discussing sequences or series. That principal speaks to the convergence of sequences. It becomes elaborate when applied to series. The principal needed deals with sequences that maintain their direction. These are sequences that get steadily larger or steadily smaller. Such sequences are called "monotonic.

> Any monotonic decreasing sequence that is bounded below *must* converge.Any monotonic ascending sequence that is bounded above *must* converge.

The principal speaks to two distinct characteristics of a sequence. It must be monotone and it must be bounded in the correct direction.

NOTE: Remember, this principal only applies to sequences. The sequence of partial sums we discussed is a decimal expansion but we have no formula for the n^{th} entry in order to test the limit of the sequence. The remaining tests in our textbook, which apply *only* to infinite series, are dependent on this basic principal.

(WORK EXERCISES 4)

V. Geometric Series Applications

There are many practical problems modeled by the geometric series. Recall, each term of a geometric series is a constant multiple of the previous term. From our previous discussion, we know if the sequence of partial sums generated by this series converges, then the series converges. In the geometric series the behavior of r^n, the constant ratio, as $n \to \infty$ determines the behavior of the geometric series.

> Geometric Series
>
> Let r and a be real numbers. If $|r| < 1$, then $\sum_{n=1}^{\infty} a\,r^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.

(WORK EXERCISES 5 and 6)

Math 112 S24 Lab 7 Exercises   Name: ____________________________ Section:_______ Score: ____

Read through this lab and work the corresponding problems. Students may work cooperatively but must submit their own set of Lab Exercises. **No calculators.**

1.Consider the *sequence* $\{1, 3, 5, ...\}$ (a) Write a simple verbal description of this sequence. (b) Find a formula for the n^{th} entry in this list.

2. (a) Write a few entries in the sequence of successive decimal approximations for 1/3.

(b) Write a formula for the n^{th} entry in this list. (Hint: .9 = 1 – 1/10, .99 = 1 – 1/100, .999 = 1 – 1/1000. What can you do to these values to yield the sequence in 2(a)?)

3. (a) Find the first 5 terms of each *sequence* defined below:

$$a_n = \frac{1}{n} =$$

$$b_n = \frac{1}{n^2} =$$

(b) What can be said about the convergence of each *sequence*? Justify your answer by showing limits.

(c) Find the first 5 partial sums for each related series and write them as a *sequence of partial sums,* S_n .

i.) $\sum_{n=1}^{\infty} \frac{1}{n}$

ii.) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(d) Write a statement concerning the long-term behavior of each *sequence of sums* in (c).

i.)

ii.)

4. (a) Write out the first 10 entries in the *sequence* of decimal additive "corrections" to π. The start value will be 3.

(b) Now, write out the corresponding *sequence of partial sums* of those decimal corrections.

(c) Which of these two sequences converge?

(d) Fill in the blank: Decimal expansions ___________________ (always, sometimes, never) converge.

5. The Bouncing Ball (geometric series problem)

A certain ball has the property that each time it falls onto a hard, level surface it rebounds to 75% of its previous height. Suppose this ball is dropped from a height of 10 feet above the floor.

(a) Sketch the ball's vertical travel for several bounces. Vertical is ball height, horizontal is bounce on the floor.

(b) How far has the ball traveled when it touches the floor. (Consider up and down as positive distance.)

i.) the first time?

ii.) the second time?

iii.) the third time?

(c) Suppose the ball continues to bounce indefinitely. Use the information in (b) to find the proper formula for the sum, then find the total distance traveled by the ball. (Hint: The total distance may be written as a single value plus a geometric series.)

6. When money is spent on goods and services, the people who receive the money also spend some of it. Those who receive the twice-spent money will spend some of that, and so on. Economists call this chain reaction the "multiplier effect." Suppose the IRS sends each taxpayer a rebate of $300 and that each recipient spends 90% of the money received (this is called the "marginal propensity to consume.")

(a) Fill in the table that shows movement of the money through five rounds of spending. ($300 is a_0)

n Number of rounds of spending	a_n Amount spent on each round	S_n Cumulative sum of spending through this round
0		
1		
2		
3		
4		
5		

(b) Assume this spending pattern continues. Write a geometric series that models the amount of money that moves through the economy beginning with the $300 payment.

(c) Calculate the total amount of spending that is generated by the $300 payment.

(d) Economists call the ratio of the total spending to the original payment the "multiplier." Calculate the multiplier for the example above. This tells you how many times the original payment has "turned over."

(e) If the marginal propensity to consume is 95%,

i.) Calculate the total amount of spending that is generated by the $300 payment. Show how you know.

ii.) Find the multiplier.

(f) How does this idea of the "multiplier effect" justify the government giving such a rebate to taxpayers?

Helpful Calculator Stuff:

We can get a look at the values of both the

sequence $a_n = \dfrac{1}{n}$ and its related series $\sum_{n=1}^{\infty} \dfrac{1}{n}$

Clear y= and lists: **2nd** , **MEM**, **4**, **enter**

Bring up table as shown: **STAT**, **ENTER**

Arrow over and/or up to highlight L_1

2nd, **LIST**, **OPS**, **seq(x,x,1,100,1)**, **ENTER**

This gives us the #s in L_1

Arrow right and up to highlight L_2. Type:

L_1	L_2 terms	L_3 partial sums
1		
2		
3		
4		
5		
6		
7		

1 ÷ **2nd** L_1, **ENTER** This gives you the terms in the sequence $a_n = \dfrac{1}{n}$

Arrow right and up to highlight L_3. Type: **2nd**, **LIST**, **OPS**, **6**, **ENTER**, **2nd**, L_2, **ENTER**