Math 112
Lab 5

Approximating Definite Integrals

I. Introduction

We have studied several methods that allow us to find the exact values of definite integrals. However, there are some cases in which it is not possible to evaluate a definite integral exactly. In evaluating $\int_a^b f(x)dx$ using the Fundamental Theorem of Calculus we must have an antiderivative for the integrand $f(x)$. In some cases it is difficult, or even impossible, to find such an antiderivative. In such cases, an approximate (or numerical) integration technique can be employed. These approximation methods are also used in many application problems where there is no explicit formula for the function of interest.

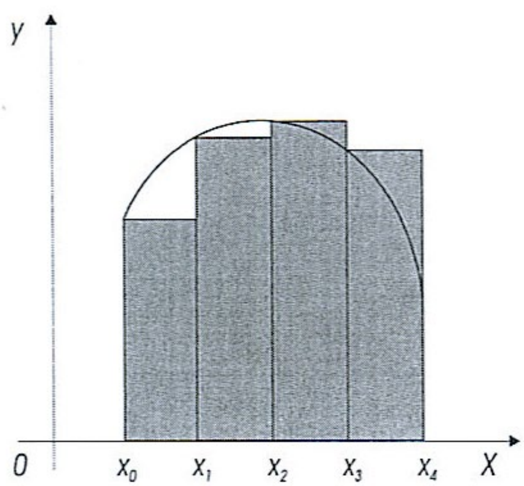
II. Left- and Right-hand Riemann Sums $(L_n \text{ and } R_n)$

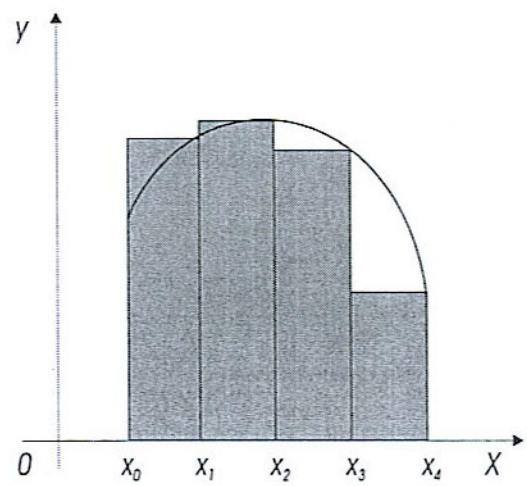
In Calculus I you learned an approximation method using a Riemann sum. Recall that the Riemann sum is formed by dividing the interval [a, b] into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0, x_1, x_2, \ldots, x_n$ be the endpoints of the subintervals and choose sample points $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ in these subintervals, so that x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. The definite integral is then approximated as

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

Taking the left- or right-hand endpoints of each subinterval as sample points is a simple approach in formulating a Riemann sum. In doing so, we obtain the following approximations:

$$\int_a^b f(x)dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \int_a^b f(x)dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$





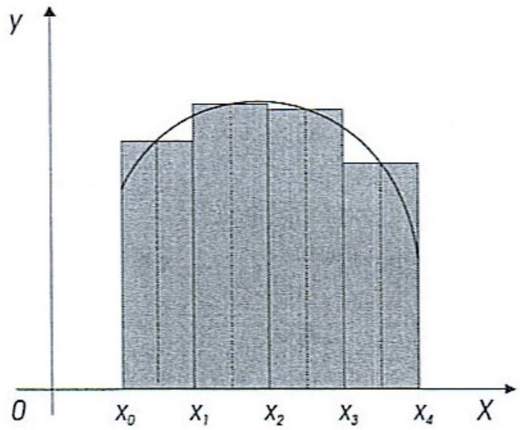
The left- and right-hand sums approximate the area under the graph as the sum of the areas of rectangles having width Δx and height $f(x_{i-1})$ or $f(x_i)$, respectively, as shown above. Notice the left-hand sums yield a low estimate of the area under $f(x)$ when the function is increasing then a high estimate of the area when the function is decreasing. The opposite occurs for right-hand sums.

III. The Midpoint Rule (M_n)

Another commonly used sample point for a Riemann sum approximation is the midpoint of each subinterval, which is denoted $\overline{x}_i$. Here the Riemann sum is formed by dividing the interval [a, b] into *n* subintervals of equal width $\Delta x = \dfrac{b-a}{n}$ and $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$ is used to obtain the height of each rectangle. The approximation is:

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$$

The midpoint rule approximates the area under the graph of $f(x)$ from *a* to *b* as the sum of the areas of rectangles having width Δx and height $f(\overline{x}_i)$ as seen in the figure to the right. Notice from the figure that the midpoint rule provides a better approximation to the definite integral than the left- or right-hand sums with the same number of subdivisions, *n*.

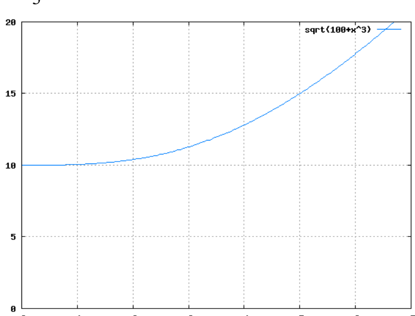


Example 1: For the integral $\int_2^5 \sqrt{100 + x^3}\, dx$,

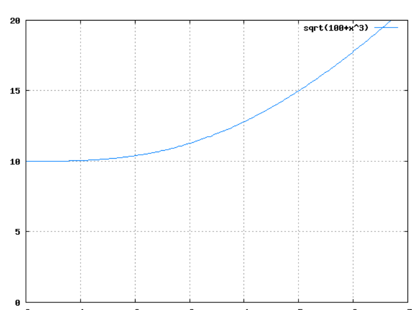
(a) calculate the approximation M_3. Round your answer to four decimal places.

(b) sketch the rectangles corresponding to $L_3, R_3, and\ M_3$. Which approximation is an overestimate?

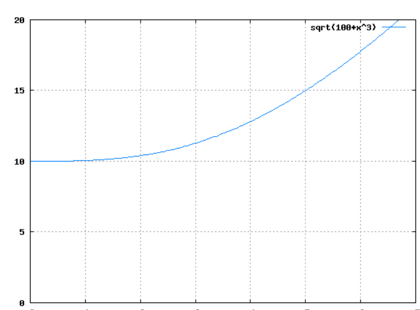
i) L_3



ii) R_3



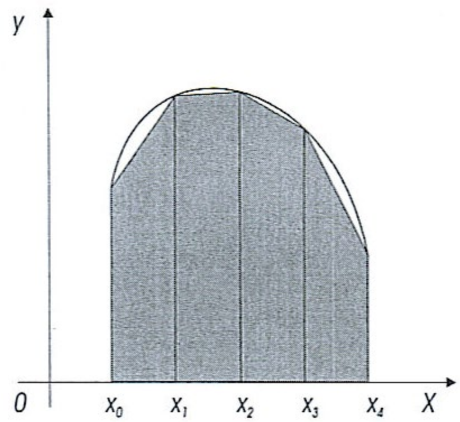
iii) M_3



IV. The Trapezoid Rule (T_n)

The previous methods of approximation use the sum of the areas of rectangles. The areas of trapezoids may also be summed to approximate a definite integral. In this case, the interval [a, b] is, once again, divided into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0, x_1, x_2, ..., x_n$ be the endpoints of the subintervals. The definite integral is then approximated as

$\int_a^b f(x)dx \approx T_n = \sum_{i=1}^{n} \frac{(f(x_{i-1}) + f(x_i))}{2} \Delta x$ and is depicted below.



We are familiar with the formula for the area of a trapezoid: $A = \frac{1}{2}h(a+b)$ where h is the height and a and b are the lengths of the each side. In the approximation formula above $h = \Delta x, a = f(x_{i-1}), and\ b = f(x_i)$ for each trapezoid under $f(x)$. The summation formula above also reveals another important property of the trapezoid rule. It is the average of the left- and right-hand sums.

That is, $T_n = \frac{L_n + R_n}{2}$

The formula used for the Trapezoid Rule is $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]$.

V. Simpson's Rule (S_n)

Simpson's rule differs from the other methods in that *it uses parabolas instead of straight-line segments* to approximate the area under the function over each subinterval. In this case, the interval [a, b] is again divided into n subintervals (but now n must be even) of equal width $\Delta x = \frac{b-a}{n}$ and $x_0, x_1, x_2, ..., x_n$ are the endpoints of the subintervals. Then on each consecutive pair of subintervals the function is approximated by a parabola that is forced to match the value of $f(x)$ at the two endpoints and the midpoint of the subinterval. The definite integral is then approximated:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

It is interesting to note that the approximation using Simpson's Rule has a weighted average relationship with the Midpoint and Trapezoid rules as follows:

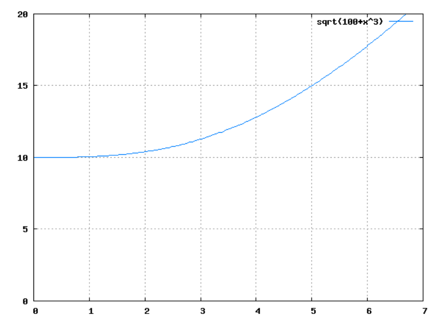
$$S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$$

*Note: When using any of these approximation methods on raw data such as that found in a table, use exact table values. Do not interpolate.

Example 2: For the integral $\int_2^5 \sqrt{100+x^3}\,dx$,

(a) calculate the approximation T_3 using the formula above.

(b) sketch the trapezoids used in the T_3 approximation. Does T_3 yield a *low* or *high* estimate?



Example 3:

(a) Approximate the definite integral $\int_2^5 \sqrt{100+x^3}\,dx$ using Simpson's rule with $n = 6$.

(b) Show this approximation satisfies the weighted average equation above using M_3 and T_3 found previously.

VI. Approximating Errors

In applying approximation techniques, error is involved (in almost every case.) The exact error involved is the difference in the actual value of the definite integral and the approximate value. That is, if A_n is some approximate value obtained using *n* subintervals, then $Error = \left|\int_a^b f(x)dx - A_n\right|$. Generally, increasing *n* improves the accuracy of the approximation. This leads to the question of how large to take *n*. There are some error bound formulas that allow us to determine the greatest expected error for a desired *n*. Of course, given a maximum allowable error, one could also determine the smallest *n* that would yield such an error. The errors for the Trapezoid and Midpoint Rules, respectively, are:

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \text{ and } |E_M| \le \frac{K(b-a)^3}{24n^2} \text{ where } |f''(x)| \le K \text{ for } a \le x \le b$$

Here, *K* is the greatest extrema (in absolute value) of the second derivative of the integrand on the interval [a, b].

Simpson's Rule is the most accurate approximation method with the maximum error $|E_S| \le \frac{K(b-a)^5}{180n^4}$, where *K* is the greatest extrema (in absolute value) of the *fourth derivative* of the integrand on [a, b].

Example 4: Suppose the integral $\int_0^1 e^{x^2}\,dx$ is approximated using either the Trapezoid or Midpoint rule.

(a) Explain why K = 18 might be used for the error bound formula as it applies to $\int_0^1 e^{x^2}\,dx$. (Hint: $e \approx 3$)

(b) Determine the maximum error incurred in approximating $\int_0^1 e^{x^2}\,dx$ using M_4.

(c) If the maximum error tolerance in using T_n to approximate $\int_0^1 e^{x^2}\,dx$ is 0.001, what is the smallest *n* that should be used?

Math 112 S24 Lab 5 Exercises Name: ____________________________ Section: _____Score: ____

You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. No calculators unless noted otherwise.

1. Consider the definite integral, $\int_1^7 \sqrt{x^3+1}\, dx$.

(a) Use the Midpoint Rule with $n = 6$ to approximate this integral. Show all intermediate steps but use your calculator for final computation (you can check your work using the "Helpful Calculator Stuff" on 5-9.) Retain at least three decimal places.

(b) Use Simpson's Rule with $n = 6$ to approximate this integral. Show all intermediate steps but use your calculator for final computation. Retain at least three decimal places.

(c) Which approximation is most likely to be more accurate? Explain why.

2. Consider the definite integral, $\int_0^1 xe^{-x}\,dx$.

(a) Approximate the integral using T_4 . Show all intermediate steps but use a calculator for final computation and retain at least 3 decimal places using proper rounding.

(b) Calculate the *exact* value of $\int_0^1 xe^{-x}\,dx$. No calculator; final answer should be an exact value.

(c) Compute the ***actual error*** , $|E_T|$, in approximating this integral using T_4. Use a calculator; retain 3 decimal places.

(d) Calculate the ***maximum expected error*** , $\frac{K(b-a)^3}{12n^2}$, in using T_4.

i.) Find $f''(x)$ then graph it on your calculator to determine the value of *K* (explain your reasoning).

ii.) Calculate the ***maximum expected error*** in using T_4 to approximate the value of this integral.

iii.) Check to see if your ***actual error*** in part (c) be less than or equal to $|E_T|$?

L_1 x-values	L_2 terms	L_3 cumulative sums
…	…	…

Helpful Calculator Stuff:

Most scientific calculators have the capacity to create and stores lists of numbers and quickly sum these. The instructions below are for a TI-84 or similar TI calculator.

1. Clear y= and lists: **2nd** , **MEM**, **4**, **enter**
2. Bring up table as shown: **STAT**, **ENTER (for 1, EDIT)**
3. In list L_1 enter the *x*-values you wish to insert into a function.
4. Arrow over and up to highlight L_2 : Type function, replacing variable *x* with L_1 (using 2nd, press "1")
5. This will display the function value at each *x*-value listed in L_1
6. Arrow right and up to highlight L_3 . Enter: **2nd**, **LIST**, **OPS**, **6**, **2nd**, L_2 **, ENTER**
7. This column contains the cumulative sums so the last number in this column is the sum of the terms of the Reimann Sum.