The College Of WILLIAM & MARY

Math 112
Lab 4

The Method of Partial Fractions

I. Introduction

Another integration technique is called the *method of partial fractions*. This technique is used to find the antiderivatives for a certain class of functions, the rational functions. Recall, a rational function is a function of the form $f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions. For example:

$\displaystyle\int \frac{6x^3-3x^2+7x-1}{2x^2-x-1}dx$ has a rational function for its integrand. There is no obvious function whose derivative is this rational function. If we can rewrite such a rational function as the sum of simpler rational functions that are easier to integrate, we have a method for evaluating the integral. Simplifying a rational function to the sum of simpler rational functions is called *partial fraction decomposition*. The general steps to evaluating these types of integrals are:

1. Determine if the integrand is a *proper* rational function (deg $p(x)$ < deg $q(x)$). If it is not, use long division to rewrite the integrand as $\dfrac{p(x)}{q(x)}=Q(x)+\dfrac{r(x)}{q(x)}$ where $Q(x)$ is a quotient and $r(x)$ a remainder.

2. Factor the denominator, $q(x)$, of the proper rational function and determine the proper form to use for the partial fraction decomposition:

(a) If $q(x)$ contains **unique linear factors** use $\dfrac{p(x)}{q(x)}=\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{a_2x+b_2}+\ldots+\dfrac{A_n}{a_nx+b_n}$

(b) If $q(x)$ contains **repeated linear factors** use $\dfrac{p(x)}{q(x)}=\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{(a_1x+b_1)^2}+\ldots+\dfrac{A_n}{(a_1x+b_1)^n}$

(c) If $q(x)$ contains **irreducible quadratic factors** use

$$\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\frac{A_2x+B_2}{a_2x^2+b_2x+c_2}+\ldots+\frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$$

(d) If $q(x)$ contains **repeated quadratic factors** use

$$\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2}+\ldots+\frac{A_nx+B_n}{(a_1x^2+b_1x+c_1)^n}$$

Note: $q(x)$ is likely to contain mixed factors that will require a combination of these forms.

3. Equate the proper rational function $\dfrac{p(x)}{q(x)}$ with the form obtained above. Solve for the constants.

4. Rewrite the integrand, $\dfrac{p(x)}{q(x)}$, in the decomposed form obtained and solve the integral.

Example 1:

Evaluate $\int \frac{10x-1}{2x^2-x-1}\,dx$

1. Is degree $p(x)$ < degree $q(x)$? (If not, use long division first.)

2. Partial Fraction Decomposition:
(a) Factor $q(x)$. Based on the factors obtained, set up the needed form(s).
(b) Multiply both sides by the least common denominator.
(c) Simplify right side and collect coefficients of like terms.
(d) Equate coefficients on left and right sides to form a system.
(e) Solve the system for A, B, etc.

3. Integrate the decomposed form of $\frac{p(x)}{q(x)}$.

Example 2: For each rational expression below, set up the proper form for partial fraction decomposition. DO NOT solve for the unknown constants.

(a) $\dfrac{x}{(x+3)^2}$

(b) $\dfrac{x-5}{x^3+x^2}$

(c) $\dfrac{2}{x^3+5x^2+8x}$

(d) $\dfrac{x^3}{x^2+1}$

Example 3: Solve $\displaystyle\int \frac{10}{(x-1)^2\left(x^2+9\right)}\,dx$

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You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. **No calculators.**

A. For each rational function, perform steps 1, 2, and 3 to set up the partial fraction decomposition of the integrand. DO NOT solve for the unknown constants.

1. $\dfrac{5x^2-3x+2}{x^3-4x}$

2. $\dfrac{1}{x^4-10x^2+9}$

3. $\dfrac{x^2}{(x-1)(x^2+1)^2}$

B. For each integral, perform steps 1-4 to evaluate. Show all your work, identify substitutions.

1. $\displaystyle\int \frac{12x-1}{x^2-5x+6}\,dx$

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2. $\int \frac{2x^2 + 7x + 4}{x^2 + 2x}\,dx$

3. $\int \frac{10}{(x-1)^2 (x^2+9)} \, dx$