Applications of Integration

In Lab 2 we explored one application of integration, that of finding the volume of a solid. Here, we explore a few more of the many applications of the definite integral by solving problems in areas such as physics, business and biology. There are many situations in which the quantity of interest may be expressed as a definite integral. Most of these follow the general procedure of dividing the quantity of interest into small pieces, solving the problem approximately for each small piece, and then summing the resulting approximations. Summing the areas under the curve of a function over each subinterval yields a Riemann sum, and the definite integral is the limit of the Riemann sum as the subintervals become smaller and smaller. In many problems however, the difficulty lies in recognizing the quantity that we want as a Riemann sum.

I. Introduction

The basic goal is to calculate a variety of quantities by setting up and evaluating a definite integral. The general procedure may be outlined as follows:

Step 1: Chop up the desired quantity into very thin slices.

Step 2: Within each slice, calculate an approximation to the desired quantity.

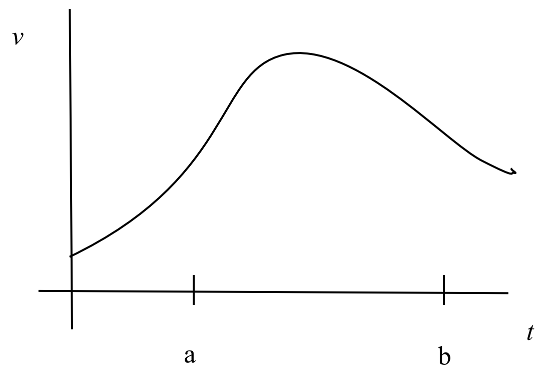
Step 3: Add up the results of all of the slice approximations from Step 2. The resulting sum is a Riemann sum that approximates the desired total quantity.

Step 4: Obtain a definite integral by taking the limit of the Riemann sum in Step 3 as the slices get thinner and thinner.

Example 1

An object travels in a straight line with velocity $v(t)$ at time *t*. From our work in 5.1, we know the area under a velocity curve over some interval of time represents the distance traveled in that time. We are taught to integrate the velocity function over this time interval to get the exact distance traveled. Here, we show how this integral is established by first creating a Reimann sum. To find the net distance traveled in the time interval $a \le t \le b$ we first slice the time interval into n subintervals of length.

a) Given a velocity curve, shade the area under this curve from $t = a$ to $t = b$. Divide this interval into several subintervals of uniform width and label the width Δt. Select any single subinterval and mark the midpoint, t_i, where $i = 1, \ldots, n$. Use the midpoint of this subinterval to find the height $f(t_i)$ then sketch this representative rectangle (width of Δt and height, $f(t_i)$). We assume Δt is small and that we can approximate the distance traveled during this small amount of time by taking the velocity to be constant over Δt. Now, write the formula that represents the approximate distance traveled over any one of these small subintervals of time.



b) Summing the approximate distances traveled in each subinterval leads to a Riemann sum that approximates the net distance traveled during the time $a \le t \le b$. Based on your answer in part a, write a Riemann sum that approximates the distance traveled during the total time interval. (Hint: this result will look like a right-hand sum approximation.)

c) The final step is to obtain a better and better approximation by taking smaller and smaller Δt. Write the limit of the Riemann sum in part b as $\Delta t \to 0$, to obtain the definite integral that represents the net distance traveled.

II. Work

In physics, the term work has a meaning that differs from its everyday usage. Physicists say that if a constant force (i.e. a push or a pull) is applied to some object to move it some distance then the force has done work on the object. The definition of work may be written as

$$W = F \cdot d, \text{ where } W \text{ is work done, } F \text{ is force and } d \text{ is distance.}$$

Notice that according to this definition, if the object does not move, then no work is done. So, if we hold a book at the same height for some time, no work is done. A force must be exerted to keep the book from falling (to counteract the force of gravity), but the book is not moved through a distance. Additionally, if we walked across the room while holding the book at the constant height, we would not accomplish any work on the book, since the force that we exerted is in the vertical direction and the movement of the book was in the horizontal direction. In other words, the direction of motion must have a component in the direction of the applied force for work to be performed.

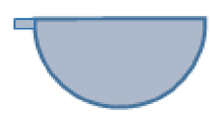
The definition provided above defines work as long as the force is constant. If we are faced with a problem in which the force is variable, we need to make some provisions. However, we already know a procedure for getting around this problem, namely slicing and summing. Faced with a variable force, we slice the problem in such a way that the force may be assumed constant over each piece. We then calculate the work for each slice using $W = Fd$, and then sum the pieces to approximate the total work.

In the next example both force and the distance are variable. We also need to use some geometry to find the formula for force. It is important to identify the location of zero for each problem.

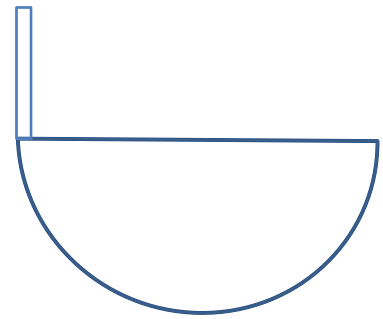
Example 2

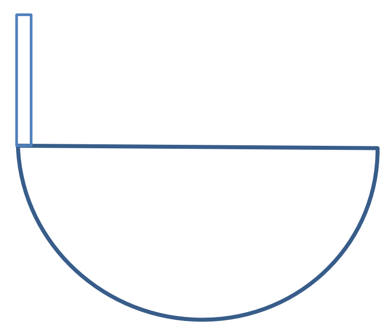
Water is to be pumped from a hemispherical water tank. The tank has a radius of 5 meters and is initially full of water. Water has density $1000\,\frac{kg}{m^3}$.

(a) Find the work to pump all of the water up and out of the top of the tank.



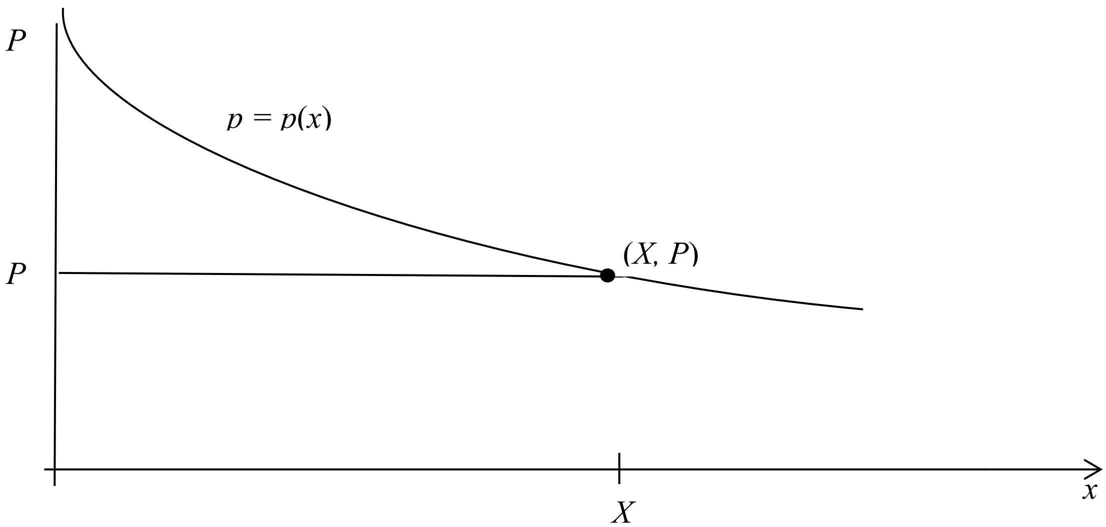
(b) Suppose the tank initially contains water to a depth of 4 meters. Find the work to pump water to a point 4 meters above the tank leaving a depth of 2 meter of water.





III. Applications in Economics

A **demand function**, $p(x)$, gives the price a company must charge in order to sell x units of a commodity. The graph of the demand function, called a **demand curve**, is decreasing since usually selling larger quantities requires lowering prices. The area between the demand curve and the horizontal line $p = p(X)$, where X is some fixed number of units, is called the **consumer surplus** for the commodity. It represents the amount of money saved by consumers in purchasing the commodity at price $p = p(X)$.



Example 3

The demand function for a certain commodity is $p = 20 - .05x$. Find the consumer surplus when the sales level is $X = 300$.

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You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. Check your computations using a calculator. *Include the proper unit in each answer.*

1.A 3-meter chain is hanging straight down the side of a building as shown at the bottom of the page. This chain has a variable density of $p = x^2 - 3x + 10$ in kg/m . Acceleration due to gravity is $9.8\, m/s^2$. We are interested in the work to pull all the chain to the top of the building.

(a) Label the sketch with the location of $x = 0$. *Your choice for the location of zero must be used for the remainder of the problem.*

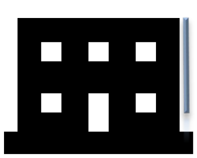
(b) Write an expression for $F(x_i)$, the force acting on any small interval of chain.

(c) Find the expression for the distance any representative part of the chain must travel (distance in terms of x_i). (The exact expression will depend on your location of zero in (a).)

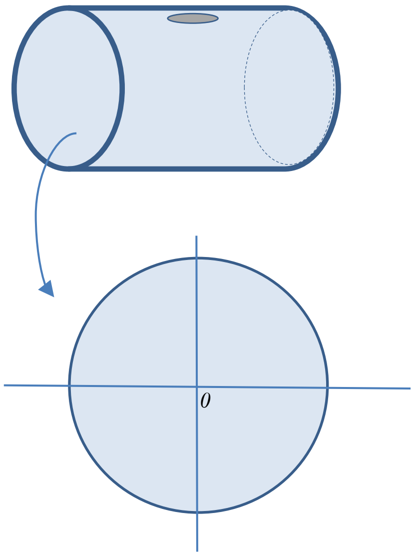
(d) Write the expression for $W(x_i)$, the work to raise any small representative part of the chain.

(e) Set up (but do not solve) the Reimann sum that approximates the total work done in lifting all of the chain.

(f) Set up and solve the proper definite integral to find the total work done in lifting all the chain to the top of the building. Include all steps of integration and include the proper work unit in your final answer.



2. A cylindrical tank contains oil (use density as ρ in lb/ft^3). The radius of the tank is 3 feet, the length is 12 feet and oil enters and leaves the tank through an opening at the top. The tank is initially full of oil. Use the sketch below and complete each step below to find the work done in pumping all of the oil out of the opening at the top of the tank. *You must use the axis provided.* **No calculators.**



(a) Find and expression for the volume of a single representative "slab" of oil that will move out of the tank.

(b) Find the expression for the distance any single representative "slab" of oil must move to get out of the tank.

(c) Find the expression for the force of a single representative "slab" of oil.

(d) Set up (BUT DO NOT SOLVE), the Reimann sum that approximates the total work done in pumping all of the oil out of the tank.

2. (cont.)

(e) Set up and solve the definite integral to find the total work done in pumping all of the oil out of the tank. Show your method(s) of evaluating the integral. Hint: You should split this into two integrals and notice one can be evaluated by interpreting it in terms of area as taught in 5.2. Keep exact values and attach the proper work unit in your final answer.

3. Use the same cylindrical tank as in problem 2. If the tank is initially full of oil, set up (BUT DO NOT SOLVE) the definite integral to find the work to pump oil out of the tank leaving a depth of 2 feet.