The College of WILLIAM & MARY

Math 112 Spring 2012
Lab 10

Name:______________________________ Date:______________

Professor:______________________________ Section:____________

Meet the Maclaurin Series

Summary: In the final weeks of class, you will be introduced to the notion of a *Taylor Series*, which extends the idea of local linearization from Math 111.

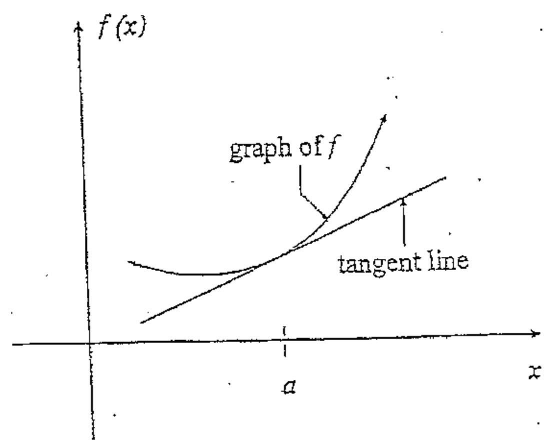
Calculator Skills: We will make use of the general graphing capabilities of your calculator.

1. Introduction: Recall the following formula from Math 111.

Given a function $f(x)$, we have the *linear approximation formula*: if $x \approx a$, then

$$f(x) \approx f(a) + f'(a)(x-a).$$

Of course, we've used this formula throughout Math 112, most notably when we applied Euler's Method to differential equations. The basic idea behind linear approximation formulas is that, near $x = a$, the graph of f and the graph of the tangent to f at $x = a$ are nearly identical. Therefore, near $x = a$, their equations must also be identical.



1. **Linear Approximations:** Consider $f(x) = e^x$.

(a) Find the linear approximation formula for $f(x) = e^x$ for $x \approx 0$.

(b) Use your calculator to graph both $f(x) = e^x$ and the tangent line from part (a). Sketch the resulting graphs.

Now, suppose instead of using a tangent *line* to approximate a function, we used a tangent *parabola*. Since the equation of a parabola is a second degree polynomial, we want a formula of the type

$$f(x) \approx a_0 + a_1 x + a_2 x^2,$$

where a_0, a_1, a_2, are constants. (Such an approximation is called a quadratic approximation formula.) How do we find such an approximation? In other words, how do we find the constants a_0, a_1, a_2 so that the above approximation is true?

To make life simpler, let's suppose that we want the approximation formula for $x \approx 0$. We first note:

The *linear* approximation formula satisfies the following conditions:

- *Condition 1:* At $x = 0, f(x)$ and the tangent line meet.
- *Condition 2:* At $x = 0, f(x)$ and the tangent line have the same slope (*i.e.* their derivatives are equal at $x = 0$.

In an analogous fashion, for the *quadratic* approximation formula, we have the following

> The *quadratic* approximation formula satisfies the following conditions:
>
> - *Condition 1:* At $x = 0$, $f(x)$ and the tangent parabola meet.
> - *Condition 2:* At $x = 0$, $f(x)$ and the tangent parabola have the same slope (*i.e.* their derivatives are equal at $x = 0$.
> - *Condition 3:* At $x = 0$, $f(x)$ and the tangent parabola have the same concavity, (*i.e.* their second derivatives are equal at $x = 0$.

2. Quadratic Approximations: Given $f(x)$, suppose we want the *quadratic* approximation formula.

$$f(x) \approx a_0 + a_1 x + a_2 x^2,$$

for $x \approx 0$. Show that if we impose the 3 conditions above, then

$$a_0 = f(0), \qquad a_1 = f'(0), \qquad a_2 = \frac{f''(0)}{2}.$$

(Hints: To make life easier, let $p(x) = a_0 + a_1 x + a_2 x^2$ be the equation of the tangent parabola. The three stated conditions amount to saying that $f(0) = p(0)$, $f'(0) = p'(0)$, and $f''(0) = p''(0)$. Use these three equations to obtain the formulas for a_0, a_1, a_2.)

3. Suppose $f(x) = e^x$.

(a) Using your result from problem 2, find the equation of the tangent parabola at $x = 0$. Write the resulting quadratic approximation formula for f when $x \approx 0$.

(b) Use your calculator to graph $f(x) = e^x$, its tangent line from problem 1b, and its tangent parabola from part (a). Make a rough sketch of the result below.

(c) Based on the graphs of part (b), which approximation appears to be more accurate, the *linear* approximation or the *quadratic* approximation?

Of course, we need not stop at using a quadratic approximation formula. We could instead try using the tangent cubic curve, resulting in the *cubic approximation formula.*

$$f(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{for } x \approx 0.$$

Analogous to the tangent *line* and the tangent *parabola*, the tangent *cubic* must satisfy:

- *Condition 1:* At $x = 0$, $f(x)$ and the tangent cubic meet.
- *Condition 2:* At $x = 0$, $f(x)$ and the tangent cubic have the same slope (the derivatives are equal at $x = 0$)
- *Condition 3:* At $x = 0$, $f(x)$ and the tangent cubic have the same concavity, (*i.e.* their second derivatives are equal at $x = 0$.
- *Condition 4:* The third derivatives of $f(x)$ and the tangent cubic are equal at $x = 0$.

4. Cubic Approximation: Given $f(x)$, suppose we want the *cubic approximation formula*

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

for $x \approx 0$. Show that if we impose the above 4 conditions, then

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2}, \quad a_3 = \frac{f'''(0)}{6}.$$

(Hints: To make life easier, let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the equation of the tangent cubic. The three stated conditions amount to saying that $f(0) = p(0)$, $f'(0) = p'(0)$, and $f''(0) = p''(0)$, $f'''(0) = p'''(0)$. Use these equations to obtain the formulas for a_0, a_1, a_2, and a_3.)

5. Suppose $f(x) = e^x$.

(a) Using your result from problem 4, find the equation of the tangent cubic at $x = 0$. Write the resulting cubic approximation formula for f when $x \approx 0$.

(b) In your calculator, graph $f(x) = e^x$, its tangent line from problem 1b, its tangent parabola from problem 3b, and its tangent cubic from part (a).

(c) Based on the graphs of part (b), which approximation appears to be more accurate, the *linear* approximation, the *quadratic* approximation, or the *cubic* approximation?