The College Of WILLIAM & MARY

Math 112 Spring 2012
Lab 1

Exploring Substitution

I. Introduction

We use the Fundamental Theorem of Calculus, Part 2 to evaluate a definite integral. If *f* is continuous on [a, b] and *F* is any antiderivative of *f* (that is $F' = f$), then $\int_a^b f(x)\,dx = F(b) - F(a)$. But this is very limiting as we quickly run out of functions with "known" antiderivatives. How do we integrate more interesting functions such as $\int \frac{dx}{x\ln x}$ or $\int_0^2 5x\left(x^2+1\right)^6 dx$? These integrals each have a form that allows us to use the *substitution method or rule* for evaluation. The substitution method for integration is frequently used to integrate functions. It involves introducing a change in variable. This might sound as though we're creating a more complicated integral but in fact, the change in variable creates a *simpler* integral with a known antiderivative.

> The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval *I* and *f* is continuous on *I*, then
> $$\int f\left(g(x)\right)g'(x)\,dx = \int f(u)\,du$$

The integral on the left is the product of a composite function and the derivative of the "inner" part of that composite function. The substitution rule is often referred to as "the chain rule in reverse." Use the substitution, $u = g(x)$ and show (mathematically) that the integrals in the substitution rule are equal.

II. A Simple Example

We integrate $\int \sqrt{x}\,dx$ using the power formula for antidifferentiation on page 243 ($n = 1/2$.) What about $\int \sqrt{1-5x}\,dx$? Here, the integrand is a composite function with the "inner" function, $g(x) = 1-5x$. We cannot apply the power formula yet. However, let $u = 1-5x$ and find the derivative $\frac{du}{dx}$. Use the expressions for *u* and *du* to make substitutions in $\int \sqrt{1-5x}\,dx$ and create a simpler integral in *u*. Remember, the integrand often contains a *constant multiple* of derivative $\frac{du}{dx}$. Evaluate $\int \sqrt{1-5x}\,dx$

III. More Complex Substitution Example

If the integrand still has a "leftover" factor containing *x* after our *u* and *du* substitutions. Solve for *x* using the *u* substitution equation to find a substitution for this expression in *x*. Evaluate $\int x\sqrt{x+3}\,dx$

IV. Substitution Method for Definite Integral Example

Here, you have two choices. Either evaluate the corresponding indefinite integral, then use the Evaluation Theorem (FTC, Part 2) or change the limits of integration to corresponding u-values and proceed (never going back to the original variable.

> The Substitution Rule for Definite Integrals: If g' is continuous on $[a,b]$ and *f* is continuous on the range of $u = g(x)$, then $$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Evaluate $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$

V. General Steps to Substitution Method

1. Choose *u* to be the "inner" part of the composite function seen in the integrand. Identify the substitution equation $u = g(x)$.
2. Find the derivative of *u* (*du*). You might need to use the chain rule!
3. Rewrite the given integral, replacing each part of the integrand with the *u*-substitution.
   (a) Is there a "leftover" expression in *x*? If so, use $u = g(x)$ to replace *x* with an expression in *u*.
   (b) For definite integrals, change the limits of integration to *u*-values
4. The results should be a simple integral in *u*.
5. Evaluate the new integral.
6. For indefinite integrals, convert the solution to the original variable using the substitution equation.

VI. Exercises

Evaluate each integral using the substitution method. Clearly indicate the substitutions used.

1. $\int x^2 e^{x^3+1} dx$

2. $\int \sin(2-\theta)\,d\theta$

3. $\int x^2 \sqrt{x-4}\, dx$

For problems 4 and 5, change the limits of integration as in the substitution rule for definite integrals.

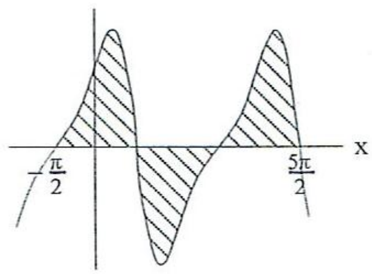
4. $\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\, dx$

5. $\int_0^{\ln 5} e^x \left(3e^x + 1\right)^{-3/2} dx$

VII. Graphical Interpretation of Substitution

In the substitution process, we take an integral written in one variable (usually *x*) and convert it to a simpler integral written in another variable (usually *u*.) This process is familiar in mathematics. It is known as a change in variable or "transformation": it is taking a problem and "mapping" it to another form. In the context of the mathematical "space" of a problem, the substitution method takes an integral written in "*x*-space" and transforms it to one written in "*u*-space."

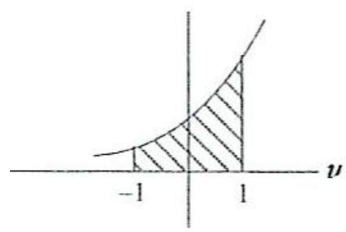
Consider the definite integral $\int_{-\pi/2}^{5\pi/2} 4e^{\sin x}\cos x\,dx$. Graphically, this definite integral represents the area under $4e^{\sin x}\cos x$ on $\left[-\frac{\pi}{2},\frac{5\pi}{2}\right]$. In *x*-space, the area looks like:



We make the substitutions $u=\sin x$ and $du=\cos x\,dx$ and then change the lower and upper limits:

$x=-\frac{\pi}{2}$,	$u=1$
$x=\frac{5\pi}{2}$,	$u=-1$

to transform the definite integral to: $\int_{-1}^{1} 4e^{u}\,du$. Graphically, this definite integral represents the area under $4e^{u}$ on $[-1,1]$. In *u*-space, the area looks like:



In mapping the problem from the x-space to the u-space, the definite integral got simpler which means the area represented by the definite integral also got simpler. The shaded areas are the same. This is easily verified using the Integral function on your calculator. Enter $Y_1 = 4e^{\sin x}\cos x$ and $Y_2 = 4e^{x}dx$ into the calculator.

Compare the values of $fnInt\left(Y_1, x, -\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and $fnInt(Y_2, x, -1, 1)$. We find that

$$\int_{-\pi/2}^{5\pi/2} 4e^{\sin x}\cos x\,dx = \int_{-1}^{1} 4e^{u}\,du \approx 9.4016$$

VIII. Exercises

6. (a) Evaluate $\int_1^e \frac{4\ln x}{x}\,dx$, show all your substitutions clearly.

(b) Sketch and shade the areas represented by both the original integral and your "transformed" integral. You may use your calculator to get the graphs but label the x- and y- axes carefully below.

(c) Use the *fnInt* function on your calculator to compare the values of the two integrals.

7. A spherical rubber ball is filled with air at a constant rate. At time, t = 0 the radius of the ball is 1 *cm*.
($V = volume$ in cm^3, $r = radius$ in *cm*, and *t*=*time* in *minutes*.)

(a) Use $V = \frac{4}{3}\pi r^3$ and find the formula that models the rate of change in the volume of the ball with respect to *radius*, $\frac{dV}{dr}$.

(b) Find the formula that models the rate of change in the volume of the ball with respect to *time*, $\frac{dV}{dt}$. (radius is also a function of time so use implicit differentiation, your formula should contain both *r* and $\frac{dr}{dt}$)

(c) The radius of the ball is increasing at a rate of $\frac{1}{2}cm$ per minute. Rewrite the rate of change formula in (b) explicitly in terms of *t* (replacing both *r* and $\frac{dr}{dt}$.)

(d) How much did the volume of the ball increase between *t* = 2 minutes and *t* = 4 minutes? Although this can be found without calculus, use integration to solve the problem.