1. Evaluate $\int x \sin(5x)\,dx$

$w = 5x \Rightarrow x = \frac{1}{5} w$

$dw = 5\,dx$

$\frac{1}{5} dw = dx$

$\frac{1}{5} \cdot \frac{1}{5} \int w \cdot \sin w \, dw$

I by P

$u = w$ $\quad dv = \sin w \, dw$

$du = dw$ $\quad v = -\cos w$

$= \frac{1}{25}\left[-w\cos w + \int \cos w \, dw \right]$

$= -\frac{1}{25}\left[5x\cos(5x) - \sin(5x) \right] + C \Rightarrow \boxed{\frac{1}{25}\sin(5x) - \frac{1}{5}x\cos(5x) + C}$

– – – – – OR – – – – – –

I by P

$u = x$ $\quad dv = \sin(5x)\,dx$

$du = dx$ $\quad v = -\frac{1}{5}\cos(5x)$

$\int x \sin(5x)\,dx = -\frac{1}{5}x\cos(5x) + \frac{1}{5}\int \cos(5x)\,dx = \boxed{\frac{1}{25}\sin(5x) - \frac{1}{5}x\cos(5x) + C}$

2. Evaluate $\int \frac{x^4 + x^2 - 1}{x^3 + x}\,dx$ (Hint: you will use long division.)

$$\begin{array}{r} x \\ x^3 + 0x^2 + x + 0 \,\big)\, x^4 + 0x^3 + x^2 + 0x - 1 \\ -\,(x^4 + 0x^3 + x^2 + 0) \\ \hline -1 \end{array}$$

$= \int x - \frac{1}{x(x^2+1)}\,dx$

PFD

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1}$

$1 = Ax^2 + A + Bx^2 + Cx$

$A + B = 0 \Rightarrow B = -1$

$C = 0$

$A = 1$

$= \frac{1}{2}x^2 - \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$

$u = x^2 + 1$ $\quad \frac{1}{2}\int \frac{1}{u}\,du$

$du = 2x\,dx$ $\quad = \frac{1}{2}\ln|u|$

$\frac{1}{2}du = x\,dx$ $\quad = \frac{1}{2}\ln(x^2+1)$

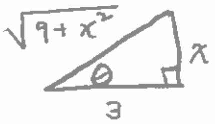
$\boxed{= \frac{1}{2}x^2 - \ln|x| + \frac{1}{2}\ln(x^2+1) + C}$

Trig Sub.

3. Evaluate $\int \frac{1}{x^2\sqrt{x^2+9}}dx$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$$= \int \frac{\cancel{3}\sec^2\theta}{9\tan^2\theta\sqrt{\cancel{9}\tan^2\theta+\cancel{9}}}\,d\theta = \frac{1}{9}\int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta\,\cancel{\sqrt{\sec^2\theta}}}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{9}\int \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\cdot\frac{1}{\cancel{\cos\theta}}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$\int u^{-2}\,du$

$= -\frac{1}{u}$

$= -\frac{1}{\sin\theta}$

$$= -\frac{1}{9}\csc\theta + C$$

$\csc\theta = \frac{hyp}{opp}$

$$= -\frac{1}{9}\cdot\frac{\sqrt{9+x^2}}{x} + C = \boxed{-\frac{\sqrt{9+x^2}}{9x} + C}$$

4. Evaluate $\int_0^{\pi/4} \sin^3(\theta)\cos^3(\theta)\,d\theta$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

when $\theta = \pi/4$, $u = \sqrt{2}/2$

$\theta = 0$, $u = 0$

$$= \int_0^{\pi/4} \sin^3(\theta)\cdot\cos^2(\theta)\cdot\cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^3\theta\,(1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \int_0^{\sqrt{2}/2} u^3(1-u^2)\,du = \int_0^{\sqrt{2}/2} u^3 - u^5\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6\Big|_0^{\sqrt{2}/2}$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)^4 - \frac{1}{6}\left(\frac{\sqrt{2}}{2}\right)^6 = \frac{1}{\cancel{4}}\left(\frac{\cancel{4}}{16}\right) - \frac{1}{6}\left(\frac{\overset{1}{\cancel{8}}}{\cancel{64}}\right)_8$$

$$= \frac{1}{16} - \frac{1}{48} = \frac{3-1}{48} = \frac{2}{48} = \boxed{\frac{1}{24}}$$

5. Evaluate $\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}dx$

$u = \arctan x$

$du = \frac{1}{1+x^2}dx$

when $x = 1, \quad u = \pi/4$

$x = 0, \quad u = 0$

$$= \int_0^{\pi/4} u^{1/2}\,du$$

$$= \frac{2}{3}u^{3/2}\Big|_0^{\pi/4}$$

$$= \frac{2}{3}\left(\frac{\pi}{4}\right)^{3/2} = \frac{\pi}{6}\sqrt{\frac{\pi}{4}}$$

6. Either find the exact value of, or show the following improper integral diverges;

$\int_{-\infty}^{1} xe^{-x^2}dx$

$$= \lim_{t\to-\infty}\int_t^1 x e^{-x^2}\,dx$$

Aside

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$-\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^{-x^2}$

$$= \lim_{t\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_t^1$$

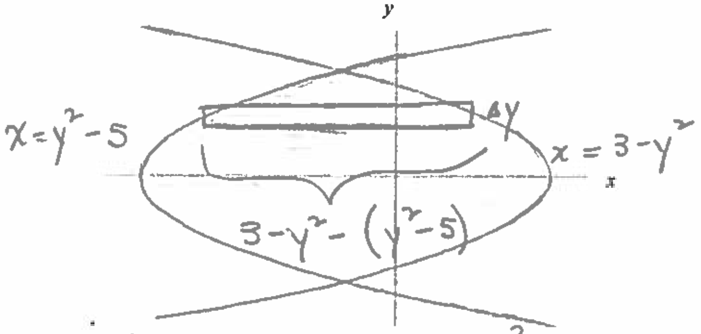
$$= -\frac{1}{2}\left[e^{-1} - \lim_{t\to-\infty} e^{-t^2}\right]$$

$$= -\frac{1}{2e} + \cancel{\frac{1}{2}\lim_{t\to-\infty}\overset{\to 0}{\frac{1}{e^{t^2}}}} = -\frac{1}{2e}$$

convergent

$x = y^2 - 5$

7. Consider the curves $\overbrace{y^2 = x + 5}$ and $x = 3 - y^2$, shown below. Set up and solve a single definite integral to find the area between the two curves.



Intersections:

$y^2 - 5 = 3 - y^2$

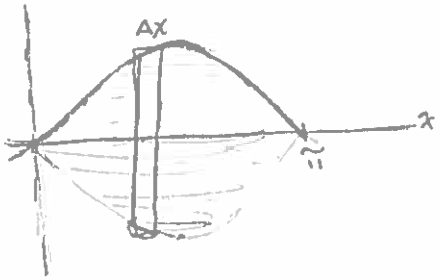
$2y^2 = 8 \quad \therefore y^2 = 4,\ y = \pm 2$

$$A = \int_{-2}^{2} 3 - y^2 - (y^2 - 5)\, dy$$

$$= \int_{-2}^{2} \underbrace{8 - 2y^2}_{\text{even}}\, dy = 2\int_{0}^{2} 8 - 2y^2\, dy = 2\left[8y - \tfrac{2}{3}y^3\right]_0^2$$

$$= 2\left[16 - \tfrac{16}{3}\right] = 2\left[\frac{48 - 16}{3}\right] = 2\left[\tfrac{32}{3}\right] = \boxed{\tfrac{64}{3}}$$

8. Set up and solve a single integral to find the volume of the solid obtained by rotating the region enclosed by $y = \sin(x), x = \pi$, and the x-axis, around the x-axis. Sketch the region and a slice of the solid.



Area of cross-sectional cut:

$A = \pi r^2$; where $r = \sin(x)$

So, $A(x) = \pi \sin^2(x)$

$$V = \pi \int_{0}^{\pi} \sin^2(x)\, dx = \frac{\pi}{2}\int_{0}^{\pi} 1 - \cos(2x)\, dx$$

$$= \frac{\pi}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_0^{\pi} = \frac{\pi}{2}\left[\pi - 0 - (0 - 0)\right]$$

$$= \boxed{\frac{\pi^2}{2}}$$

$\sqrt{}$ N.

9. A force of $f(x)=\dfrac{x}{\sqrt{x+1}}$ is applied in moving an object from a position of $x=0$ meters to $x=20$ meters in a straight line. Calculate the work done during the last 10 meters of the journey. Include the proper work units in your final answer.

$$W = \int_{10}^{20} \frac{x}{\sqrt{x+1}}\,dx$$

$u = x+1 \Rightarrow x = u-1$

$du = dx$

when $x=20$, $u=21$

$x=10$, $u=11$

$$= \int_{11}^{21} \frac{u-1}{\sqrt{u}}\,du = \int_{11}^{21} u^{1/2} - u^{-1/2}\,du$$

$$= \left.\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right|_{11}^{21} = \tfrac{2}{3}(21)^{3/2} - 2(21)^{1/2} - \left(\tfrac{2}{3}(11)^{3/2} - 2(11)^{1/2}\right)$$

Joules

10. Find the length of the curve $y=\ln(\sec x)$ from $x=0$ to $x=\dfrac{\pi}{4}$.

$y = \ln(\sec x)$

$y' = \dfrac{1}{\sec x}\cdot \sec x \cdot \tan x = \tan x$

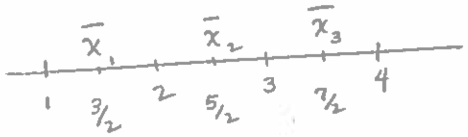
$[y']^2 = \tan^2 x$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx$$

$$= \ln\left|\sec x + \tan x\right| \Big|_0^{\pi/4}$$

$$= \ln\left|\tfrac{2}{\sqrt{2}} + 1\right| - \cancelto{0}{\ln\left|1+0\right|} = \ln\left|\frac{2}{\sqrt{2}} + 1\right| = \ln\left(\sqrt{2}+1\right)$$

11. (a) Approximate $\int_1^4 \frac{1}{x}\,dx$ using the Midpoint Rule with $n = 3$.



$\Delta x = \frac{4-1}{3} = 1$

$M_3 = 1\left[f(3/2) + f(5/2) + f(7/2)\right]$

$= 2/3 + 2/5 + 2/7 = \frac{70 + 42 + 30}{105} = \frac{142}{105}$

(b) Estimate the error in the above approximation given the error bound formula, $|E_M| \le \frac{K(b-a)^3}{24n^2}$.

Remember to show your work in finding the value of K.

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f''(1) = 2$

$f''(4) = 2/4^3$

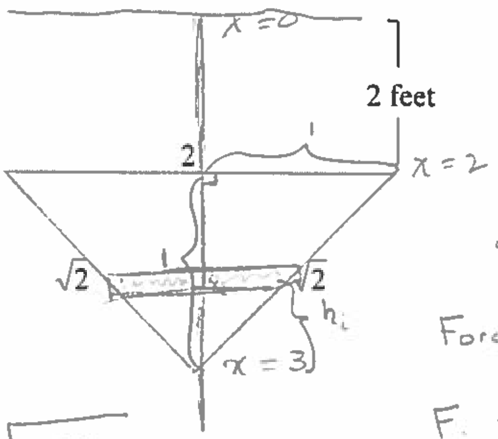
$\therefore K = 2$

$|E_M| \le \frac{\cancel{2}^{1}(4-1)^{\cancel{3}} \cancel{3}^{1}}{\cancel{24}^{\cancel{8}\,4}(\cancel{3})^{\cancel{2}}}$

$E_M \le 1/4$

$f'''(x) = -6x^{-4} < 0$ for all x $\therefore$ f'' is always decreasing (test endpoints only)

12. A plate has the shape of an isosceles triangle with two sides measuring $\sqrt{2}$ feet and the base measuring 2 feet. The plate is submerged 2 feet below the surface of a fluid with density $\delta = 90\,lb/ft^3$. Set up and solve a definite integral to find the hydrostatic force acting on one side of the plate. Remember to include the proper unit for force in your final answer.



$A_s = w_i \cdot \Delta x$ where $\frac{w_i}{h_i} = \frac{2}{1} = \frac{w_i}{3 - x_i}$

$A_s = 2(3 - x_i)\Delta x$ so, $w_i = 2(3 - x_i)$

depth, $d_i = x_i$

Force acting on a single strip is

$F_i = 2(3 - x_i)\Delta x \cdot 90 \cdot x_i$

$h = \sqrt{2-1}$

$= 1$

$F_T = 180\int_2^3 3x - x^2\,dx = 180\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_2^3$

$= 180\left[27/2 - 18/2 - (6 - 8/3)\right] = 180\left[9/2 - 10/3\right] = \cancel{180}^{30}\left(\frac{27 - 20}{\cancel{6}_{1}}\right)$

$= 210$ lbs

13. Find the solution to the differential equation that satisfies the given initial condition.

$$(1-t)\frac{dy}{dt} - y = 0, \quad y(2) = -4$$

$$\frac{dy}{dt} = \frac{y}{1-t} = \frac{(1-t)^{-1}}{y^{-1}}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1-t}\,dt$$

$$\ln|y| = -\ln|1-t| + C$$

$$\ln 4 = -\overset{0}{\cancel{\ln(1)}} + c \quad \therefore C = \ln 4$$

$$\ln|y| = -\ln|1-t| + \ln 4$$

$$y = e^{\ln\left|\frac{1}{1-t}\right|}\, e^{\ln 4}$$

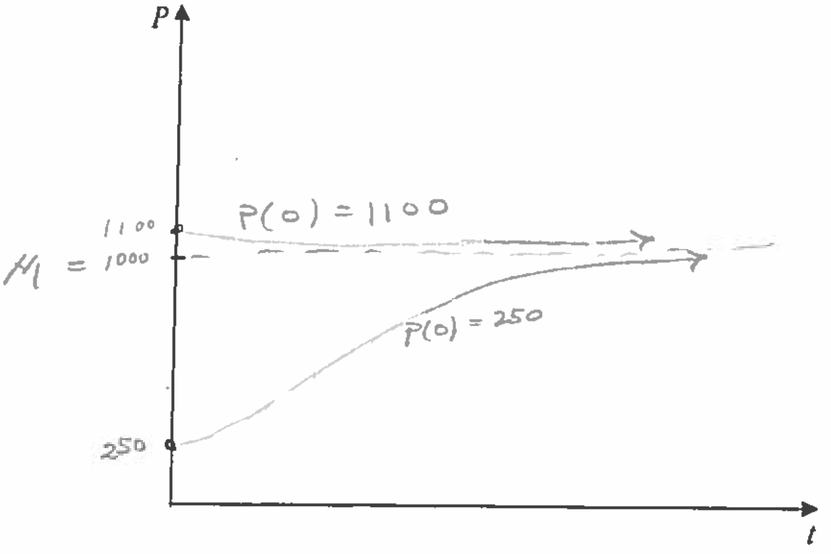
$$y(t) = 4 \cdot \frac{1}{1-t}$$

$$y(t) = \frac{4}{1-t}$$

14. Suppose some population $P = P(t)$ follows the logistic model where $\frac{dP}{dt} = 0.8P(1 - \frac{P}{1000})$.

(a) What is the carrying capacity of the population? $M = 1000$

(b) Without finding the explicit solution to the equation, draw a rough sketch of the following two solution curves; $P(0) = 1100$ and $P(0) = 250$. Label the carrying capacity and the initial values for each curve on your sketch.



15. Determine if the infinite sequence $a_n = \frac{(\ln n)}{n^2+1}$ converges or diverges. If it converges, find the limit.

$$\lim_{n\to\infty} \frac{\ln(n)}{n^2+1} \overset{\swarrow \frac{\infty}{\infty}}{\underset{Ⓛ}{=}} \lim_{n\to\infty} \frac{1/n}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

converges to 0

16. Determine if the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ converges or diverges. If it converges, find the sum. Identify the series/test(s) used and the conditions that justify each step.

PFD

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

$$2n+1 = An\overset{n^2+2n+1}{(n+1)^2} + B(n+1)^2 + Cn^2(n+1) + Dn^2$$

$$2n+1 = An^3 + 2\cancel{An^2} + \cancel{An} + Bn^2 + \cancel{2Bn} + B + \cancel{Cn^3} + \cancel{Cn^2} + \cancel{Dn^2}$$

$A+C=0$

$2A+B+C+D=0$

$A+2B=2$

$B=1 \therefore A=0$

$C=0, D=-1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \text{ (Telescoping)}$$

$$S_n = \left(1 - \cancel{\tfrac14}\right) + \left(\cancel{\tfrac14} - \cancel{\tfrac19}\right) + \left(\cancel{\tfrac19} - \cancel{\tfrac1{16}}\right) + \cdots$$

$$+ \left(\cancel{\frac{1}{(n-2)^2}} - \cancel{\frac{1}{(n-1)^2}}\right) + \left(\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}\right) + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right)$$

$$= 1 - \frac{1}{(n+1)^2} \text{ and } \lim_{n\to\infty} 1 - \frac{1}{(n+1)^2} = 1 = \text{sum}$$

converges to sum = 1

17-19. Determine if the series is *absolutely convergent*, *conditionally convergent* or *divergent* (state this). Identify the series/test(s) used and the conditions that justify each step.

17. $\sum_{n=1}^{\infty}\left|(-1)^n \frac{3^n-2}{5^n+3}\right|$ Test for abs. convergence, Comparison Test

$\frac{3^n-2}{5^n+3} < \frac{3^n}{5^n}$ and $\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$ is a geometric series with $|r| = |3/5| < 1$ $\therefore$ convergent

Series is absolutely convergent

18. $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^{1/2}}{\sqrt{n^2+2}}\right|$ Test for abs. convergence, Comparison Test

$\frac{n^{1/2}}{\sqrt{n^2+2}} < \frac{n^{1/2}}{\sqrt{n^2}} = \frac{1}{n^{1/2}}$ Inc. and $\sum \frac{1}{n^{1/2}}$ is a p-series w/ $p = 1/2 < 1$ $\therefore$ divergent

Limit Comp. Test

$\lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{n^2+2}} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+2}} = 1 > 0$ $\therefore$ both diverge

Test for conditional convergence

Alt. Series Test, $b_n = \frac{n^{1/2}}{\sqrt{n^2+2}}$

i) $f(x) = \frac{x^{1/2}}{(x^2+2)^{1/2}} = \left(\frac{x}{x^2+2}\right)^{1/2}$

$f'(x) = \frac{1}{2}\left(\frac{x}{x^2+2}\right)^{-1/2} \cdot \frac{(x^2+2)(1) - x(2x)}{(x^2+2)^2}$

$= \frac{1}{2}\sqrt{\frac{x^2+2}{x}} \cdot \frac{2-x^2}{(x^2+2)^2} < 0$ when $2-x^2 < 0$ dec

$x^2 > 2$

$\therefore x > \sqrt{2}$

ii $\lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{n^2+2}} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{1+2/n^2}} = \frac{0}{1} = 0$ ✓

conditionally convergent

✓

19. $\sum_{n=1}^{\infty} e^{-n} n^3 = \sum_{n=1}^{\infty} \frac{n^3}{e^n}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{(n+1)^3}{\cancel{e^n}\cdot e} \cdot \frac{\cancel{e^n}}{n^3} \right| = \frac{1}{e} \cdot \lim_{n\to\infty} \cancelto{1}{\left(\frac{n+1}{n}\right)^3} = 1/e < 1$

∴ Series is absolutely convergent

20. Find the power series representation for $f(x) = \frac{x^2}{1+x^3}$. Find the radius of convergence.

$f(x) = x^2 \cdot \frac{1}{1-(-x^3)} = x^2 \cdot \sum_{n=0}^{\infty} (-x^3)^n$

$= x^2 \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n+2}}$

Ratio Test:

$\lim_{n\to\infty} \frac{x^{3(n+1)+2}}{x^{3n} \cdot x^2} = \lim_{n\to\infty} \frac{\cancel{x^{3n}}}{\cancel{x^{3n}}} \cdot \frac{x^5}{x^2} = x^3 < 1$

when $x < 1$

$\boxed{R = 1}$

21. Determine the radius and interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{(-2)^n(x-3)^n}{n^2}$. Remember to show your work in testing the endpoints for the interval of convergence.

$$\sum_{n=0} \frac{(-1)^n 2^n (x-3)^n}{n^2}$$

Ratio Test: $\lim_{n\to\infty}\left|\frac{\cancel{2^n}\cdot 2\,\cancel{(x-3)^n}(x-3)}{(n+1)^2}\cdot\frac{n^2}{\cancel{2^n}\,\cancel{(x-3)^n}}\right|$

$$= 2|x-3|\cdot\lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}+1}\right)^2 = 2|x-3|<1$$

$$|x-3|<\tfrac{1}{2}$$

$$\boxed{R=\tfrac{1}{2}}$$

$$-\tfrac{1}{2}<x-3<\tfrac{1}{2}$$

$$\tfrac{5}{2}<x<\tfrac{7}{2}$$

Endpoints:

$x=5/2$: $\sum_{n=0}^{\infty}\frac{(-2)^n(-\frac{1}{2})^n}{n^2}=\sum_{n=0}^{\infty}\frac{1}{n^2}$ is p-series $p=2>1$ ∴ conv.

conv

$x=7/2$: $\sum_{n=0}^{\infty}\frac{(-2)^n(\frac{1}{2})^n}{n^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ alt. series test:

i) $\frac{1}{(n+1)^2}<\frac{1}{n^2}$ dec ✓

ii) $\lim_{n\to\infty}\frac{1}{n^2}=0$ ✓

con

Interval of Conv: $\boxed{[5/2, 7/2]}$

22. Find the Taylor series for $f(x)=e^{-3x}$ at $a=2$.

n	$f^n(x)$	$f^n(2)$
0	e^{-3x}	e^{-6}
1	$-3e^{-3x}$	$-3e^{-6}$
2	$9e^{-3x}$	$9e^{-6}$
3	$-27e^{-3x}$	$-27e^{-6}$

$$f(x)=e^{-6}-3e^{-6}(x-2)+\frac{9}{2}e^{-6}(x-2)^2-\frac{27}{3!}e^{-6}(x-2)^3\ldots$$

$$=e^{-6}-2e^{-6}(x-2)+2e^{-6}(x-2)^2-2e^{-6}(x-2)^3\ldots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{3^n e^{-6}}{n!}(x-2)^n$$