posted exam

F2012

1. Evaluate each integral below. You must show all necessary steps including substitutions, integration by parts, partial fractions, etc.

(a) $\int \sin^3(x)\cos^2(x)\,dx = \int \sin^2(x)\cos^2(x)\cdot\sin(x)\,dx$

$= \int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx$

let $u = \cos(x)$
$du = -\sin(x)dx$
$-du = \sin(x)dx$

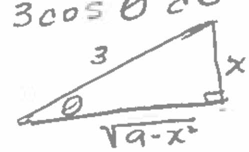
$= -\int (1-u^2)u^2\,du = -\int u^2 - u^4\,du$

$= \int u^4 - u^2\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$ ✓

(b) $\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,d\theta$

Trig. Sub.
let $x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$



$= 9\int \frac{\sin^2\theta\cdot\cos\theta}{\sqrt{\cos^2\theta}}d\theta = 9\int \sin^2\theta\,d\theta$

$= \frac{9}{2}\int 1-\cos(2\theta)\,d\theta$

$u = 2\theta$
$du = 2\,d\theta$
$\frac{1}{2}du = d\theta$
$\frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin u = \frac{1}{2}\sin 2\theta$

$= \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$

$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + C = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right] + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$

(c) $\int x\cos(3x)\,dx$

$w = 3x \iff x = \frac{w}{3}$
$dw = 3\,dx$
$\frac{1}{3}dw = dx$

$= \frac{1}{3}\int \frac{w}{3}\cos(w)\,dw = \frac{1}{9}\int w\cos(w)\,dw$ · I. by . P.

$u = w \quad dv = \cos(w)\,dw$
$du = dw \quad v = \sin(w)$

$= \frac{1}{9}\left[w\sin(w) - \int \sin(w)\,dw\right]$

$= \frac{1}{9}\left[w\sin(w) + \cos(w)\right] + C = \frac{1}{9}\left[3x\sin(3x) + \cos(3x)\right] + C$

(d) $\int \frac{2x^2 + x - 3}{x^2 - x}dx$

divide:

$$\begin{array}{r} 2 \\ x^2 - x \,\overline{)\, 2x^2 + x - 3} \\ -(2x^2 - 2x) \\ \hline 3x - 3 \end{array}$$

$= \int 2 + \frac{3x-3}{x^2 - x}\,dx$ $\quad \frac{3\cancel{(x-1)}}{x\cancel{(x-1)}}$

PFD

$\frac{3x-3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$3x - 3 = Ax - A + Bx$

$A + B = 3$
$-A = -3 \therefore A = 3, B = 0$

$= \int 2 + \frac{3}{x}\,dx$

$= 2x + 3\ln|x| + C$

(e) $\int \frac{e^x}{1+e^{2x}} dx$

$u = e^x$

$du = e^x \, dx$

$$= \int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$$

$$= \tan^{-1}(e^x) + C$$

2. Determine if the integral $\int_3^{\infty} \frac{\ln x}{x^3} dx$ is convergent or divergent. Evaluate the integral if it is convergent.

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$u = \ln x \qquad dv = x^{-3} dx$

$du = \frac{1}{x} dx \qquad v = -\frac{1}{2} x^{-2}$

$$= \lim_{t \to \infty} \int_3^t x^{-3} \cdot \ln x \, dx$$

$$= \lim_{t \to \infty} \left[-\frac{1}{2} x^{-2} \ln x + \frac{1}{2} \int x^{-3} dx \right]_3^t$$

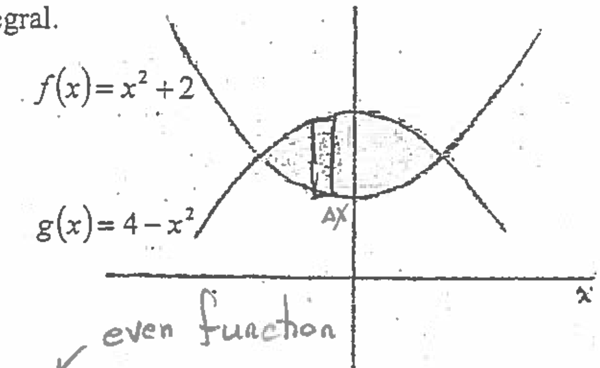
$$= \lim_{t \to \infty} \left[-\frac{1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} \right]_3^t$$

$$= -\frac{1}{2} \lim_{t \to \infty} \underbrace{\frac{\ln t}{t^2}}_{\frac{\infty}{\infty}} - \frac{1}{4} \lim_{t \to \infty} \underbrace{\frac{1}{t^2}}_{0} - \left[-\frac{\ln 3}{2(3)^2} - \frac{1}{4(3)^2} \right]$$

$$\overset{\text{L}}{=} -\frac{1}{2} \lim_{t \to \infty} \underbrace{\frac{1/t}{2t}}_{0} + \frac{\ln 3}{18} + \frac{1}{36} = \frac{2\ln(3) + 1}{36}$$

$\frac{1}{2t^2}$

3. Consider the region bound by the curves $f(x)=x^2+2$ and $g(x)=4-x^2$ that is shown below.
   (a) Find the area of this region by evaluating the proper integral.



Intersections: $x^2+2=4-x^2$

$2x^2-2=0$

$2(x+1)(x-1)=0 \therefore x=\pm 1$

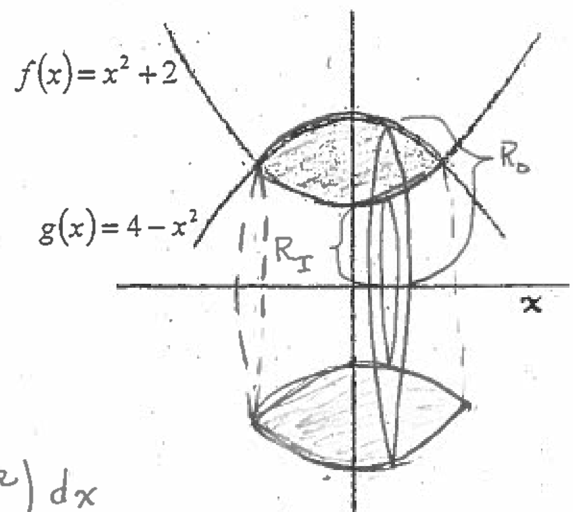
height of rep. rectangle: $4-x^2-(x^2+2)$

even function

$$A=\int_{-1}^{1} 4-x^2-(x^2+2)\,dx=\int_{-1}^{1} 2-2x^2\,dx=2\int_0^1 2-2x^2\,dx$$

$$=2\left[2x-\tfrac{2}{3}x^3\right]_0^1=2\left[2-\tfrac{2}{3}\right]=2\left(\tfrac{4}{3}\right)=\tfrac{8}{3}$$

(b) Find the volume of the solid obtained by revolving this region about the x-axis.



Area of cross-section: $A=\pi R_o^2-\pi R_I^2$

$=\pi\left(R_o^2-R_I^2\right)$

$$A=\pi\left[(4-x^2)^2-(x^2+2)^2\right]$$

$$=\pi\left[16-8x^2+\cancel{x^4}-\cancel{x^4}-4x^2-4\right]$$

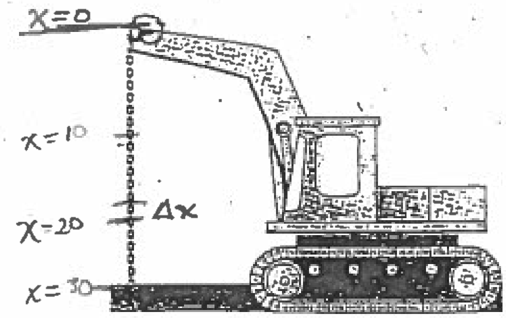
$$=\pi\left(-12x^2+12\right)$$

even function

$$V=\pi\int_{-1}^{1}(12-12x^2)\,dx=2\pi\int_0^1(12-12x^2)\,dx$$

$$=2\pi\left[12x-4x^3\right]_0^1=2\pi(12-4)=16\pi$$

4. A uniform chain that is 30 ft. long and weighs 60 lbs. is hanging vertically from a crane. Find the work required to roll up 20 ft. of the chain.



$\frac{60 \text{ lb}}{30 \text{ ft}} = 2 \text{ lb/ft}$ $W_i = F_i \cdot d_i$

Δx has $F_i = 2 \text{ lb}/\cancel{\text{ft}} \cdot \Delta x \cancel{\text{ft}}$

$= 2\Delta x \text{ lb}$ (2)

distance any Δx must travel $= x$ (2)

$d_i = x$

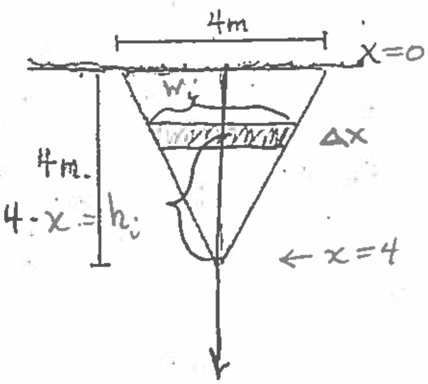
So, work to move each small Δx is $W_i = 2\Delta x \cdot x$

(2)

$$W_T = \int_{10}^{30} 2x\,dx = x^2 \Big|_{10}^{30} = 900 - 100 = 800 \text{ ft-lb}$$ (1)

(2)

5. An isosceles triangular plate is submerged vertically under the surface of a freshwater lake. The triangular plate has a 4 meter base and a 4 meter height. The base of the plate is on the water surface as shown below. The density of water is $1000\ kg/m^3$, and the gravity constant is assumed to be $10\ m/s^2$. Find the hydrostatic force acting against the plate.



$F_i = A_s \cdot \rho \cdot a \cdot \text{depth}$

$A_s = \text{width} \times \text{height} = w_i \cdot \Delta x$ $\frac{w}{4-x} = \frac{4}{4} = 1$

$= (4-x)\Delta x$ $\therefore w = 4-x$ (3)

depth $= x$ (2)

So, the force acting on any one strip is $F_i = (4-x)\Delta x \underset{1000}{\rho} \cdot \underset{10}{a} \cdot x$

$$F_T = (1000)(10)\int_0^4 (4-x)\,x\,dx = 10{,}000\int_0^4 4x - x^2\,dx$$

(1) (1) (1) (1)

$$= 10{,}000\left[2x^2 - \tfrac{1}{3}x^3\right]_0^4 = 10{,}000\left[32 - \tfrac{64}{3}\right] = 10{,}000\left(\frac{96-64}{3}\right)$$

$$= \frac{10{,}000(32)}{3} = \frac{320{,}000}{3} \text{ Newtons}$$ (1) (1)

6. Solve the differential equation $\frac{dy}{dx}=\left(x^2+1\right)\sqrt{y},\quad y(1)=1$.

$$\frac{dy}{dx} = \frac{(x^2+1)}{y^{-1/2}}$$

$$\int y^{-1/2}\,dy = \int (x^2+1)\,dx$$

$$2y^{1/2} = \tfrac{1}{3}x^3 + x + C$$

$$2 = \tfrac{1}{3} + 1 + C \quad (\text{since } y(1)=1)$$

$$2 - \tfrac{4}{3} = C$$

$$\tfrac{2}{3} = C$$

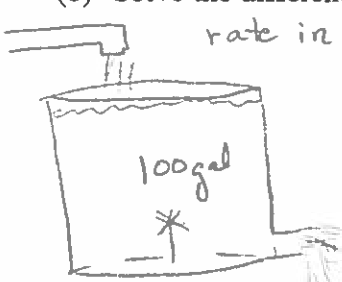
$$2y^{1/2} = \tfrac{1}{3}x^3 + x + \tfrac{2}{3}$$

$$y^{1/2} = \tfrac{1}{6}x^3 + \tfrac{1}{2}x + \tfrac{1}{3}$$

$$y = \left(\tfrac{1}{6}x^3 + \tfrac{1}{2}x + \tfrac{1}{3}\right)^2$$

7. Brine containing $1\,lb/gal$ of salt enters a tank at $2\,gal/\min$ that initially contains 100 gal of fresh water. The stirred mixture leaves the tank at $2\,gal/\min$. Let $A(t)$ represent the amount of salt in the tank at time t.
   (a) Set up the differential equation that models the rate of change of salt in the tank.
   (b) Solve the differential equation for $A(t)$.

$$A(0) = 0$$



$$\text{rate in} = \left(1\ \tfrac{lb}{gal}\right)\left(2\ \tfrac{gal}{min}\right) = 2\ lb/min$$

$$\text{rate out} = \left(\frac{A\ lb}{100\ gal}\right)\left(\frac{2\ gal}{min}\right) = \frac{A}{50}\ lb/min$$

(a) $$\frac{dA}{dt} = \text{rate in} - \text{rate out} = 2 - \frac{A}{50} = \frac{100-A}{50}$$

$$\therefore \frac{dA}{dt} = \frac{100-A}{50}$$

(b) $$\frac{dA}{dt} = \frac{50^{-1}}{(100-A)^{-1}}$$

$$\int \frac{1}{100-A}\,dA = \int \frac{1}{50}\,dt$$

$$-\ln|100-A| = t/50 + C$$

$$-\ln(100) = C \quad (A(0)=0)$$

$$-\ln|100-A| = t/50 - \ln(100)$$

$$\ln|100-A| = \ln(100) - t/50$$

$$100 - A = e^{\ln(100) - t/50}$$

$$100 - A = 100e^{-t/50}$$

$$A(t) = 100 - 100e^{-t/50}$$

– OR –

$$A(t) = 100\left(1 - e^{-t/50}\right)$$

8. Express the number $1.\overline{414} = 1.414414414\ldots$ as a ratio of two integers which have no common factors larger than 1.

$$1.\overline{414} = 1 + .414 + .000414 + .000000414 + \ldots$$

$$= 1 + \underbrace{\frac{414}{1000} + \frac{414}{1000}\left(\frac{1}{1000}\right) + \frac{414}{1000}\left(\frac{1}{1000}\right)^2 + \ldots}_{\text{geometric w/ } |r| = \left|\frac{1}{1000}\right| < 1 \ \therefore \text{ conv.}}$$

$$a = \frac{414}{1000}$$

$$\text{Sum} = \frac{a}{1-r} = \frac{414/1000}{1 - 1/1000} = \frac{414/1000}{999/1000} = \frac{414}{999}$$

$$\therefore \quad 1 + \frac{414}{999} = \frac{1413}{999}$$

9. Determine if the series $\sum_{n=1}^{\infty} \frac{e^n}{e^n - 1}$ is convergent or divergent. If convergent, find the sum.

Integral Test: $f(x) = \frac{e^x}{e^x - 1}$ is positive and continuous on $[1, \infty)$

$$f'(x) = \frac{(e^x-1)e^x - e^x(e^x)}{(e^x-1)^2} = \frac{\cancel{e^{2x}} - e^x - \cancel{e^{2x}}}{(e^x-1)^2} = \frac{-e^x}{(e^x-1)^2} < 0 \ \therefore \text{ decreasing}$$

$$\int_1^{\infty} \frac{e^x}{e^x - 1}\,dx = \lim_{t \to \infty} \int_1^t \frac{e^x}{e^x-1}\,dx$$

$u = e^x - 1$

$du = e^x\,dx$

$$\int \frac{1}{u}\,du = \ln|u| = \ln(e^x - 1)$$

$$= \lim_{t\to\infty} \ln(e^x - 1)\Big|_1^t$$

$$= \lim_{t\to\infty} \underbrace{\ln(e^t - 1)}_{\to\infty} - \ln(e-1) = \infty \quad \therefore \text{ divergent}$$

series is also divergent

T4D: $\lim_{n\to\infty} \frac{e^n}{e^n - 1} \overset{\frac{\infty}{\infty}}{\underset{\text{L}}{=}} = \lim_{n\to\infty} \frac{e^n}{e^n} = 1 \neq 0$

$\therefore$ divergent

10. Determine if the series $\sum_{n=0}^{\infty}(-1)^n \frac{(\sqrt{2})^n}{3^{2n+1}}$ is convergent or divergent. If convergent, find the sum.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{2})^n}{9^n \cdot 3} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)\left(\frac{\sqrt{2}}{9}\right)^n = \frac{1}{3} + \frac{1}{3}\left(-\frac{\sqrt{2}}{9}\right) + \frac{1}{3}\left(-\frac{\sqrt{2}}{9}\right)^2 + \frac{1}{3}\left(-\frac{\sqrt{2}}{9}\right)^3 + \cdots$$

Is geometric with $a = 1/3$, $|r| = \left|-\frac{\sqrt{2}}{9}\right| < 1$ $\therefore$ convergent

$$\text{Sum, } S = \frac{a}{1-r} = \frac{1/3}{1+\sqrt{2}/9} = \frac{1/3}{\frac{9+\sqrt{2}}{9}} = \frac{1}{\cancel{3}_{1}} \cdot \frac{\cancel{9}\,3}{9+\sqrt{2}}$$

$$\text{Sum} = \frac{3}{9+\sqrt{2}}$$

11. Determine if the series $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n+\ln n}$ is absolutely convergent, conditionally convergent or divergent.

Test for abs. conv.: $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$

(Series is NOT abs. conv.)

Comparison Test:

$\frac{1}{n+\ln n} < \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic $\therefore$ divergent (Inconcl.)

limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+\ln n}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln n}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1+\frac{1/n}{1}}$$

$\frac{\infty}{\infty}$

$= 1 > 0 \therefore$ both diverge

Alt. Series Test:

i) $\frac{1}{(n+1)+\ln(n+1)} < \frac{1}{n+\ln n}$ $\therefore$ dec. ✓

ii) $\lim_{n\to\infty} \frac{1}{n+\ln n} = 0$ ✓

$\therefore$ Series is conditionally convergent

12. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{\sqrt{n}\,3^n}$.

ratio test:

$$\lim_{n\to\infty} \left| \frac{(2x-3)^n (2x-3)}{\sqrt{n+1}} \cdot \frac{\sqrt{n}\, 3^n}{3^n \cdot 3\, (2x-3)^n} \right| = \lim_{n\to\infty} \frac{1}{3}\sqrt{\frac{n}{n+1}}\, |2x-3| = \frac{1}{3}|2x-3| < 1$$

$$= |2x-3| < 3 = 2\left|x - \tfrac{3}{2}\right| < 3 \quad \therefore \left|x - \tfrac{3}{2}\right| < \tfrac{3}{2} \qquad R = \tfrac{3}{2}$$

$$-\tfrac{3}{2} < x - \tfrac{3}{2} < \tfrac{3}{2}$$

$$0 < x < 3$$

Test Endpoints:

$x=0$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ alt series test: $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ $\therefore$ dec ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

convergent @ $x=0$

$x=3$: $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series w/ $p = \frac{1}{2} < 1$ $\therefore$ divergent

Interval of Conv. $[0, 3)$

divergent @ $x=3$

13. Evaluate the integral $\int \frac{x}{2+x^7}\,dx$ as a power series.

① Express the integrand as a power series:

$$\frac{x}{2+x^7} = x \cdot \frac{1}{2+x^7} = \frac{x}{2}\,\frac{1}{1+\frac{1}{2}x^7} = \frac{x}{2}\,\frac{1}{1-(-\frac{1}{2}x^7)} = \frac{x}{2}\sum_{n=0}^{\infty}\left(-\tfrac{1}{2}x^7\right)^n$$

$$= \frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n x^{7n} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1} x^{7n+1} = \frac{x}{2} - \frac{x^8}{4} + \frac{x^{15}}{8} - \frac{x^{22}}{16} + \dots$$

② Integrate power series:

$$\int \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1} x^{7n+1}\,dx = \int \frac{x}{2} - \frac{x^8}{4} + \frac{x^{15}}{8} - \frac{x^{22}}{16} + \dots\, dx$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}\frac{x^{7n+2}}{7n+2} = \frac{x^2}{4} - \frac{x^9}{4\cdot 9} + \frac{x^{16}}{8\cdot 16} - \frac{x^{23}}{16(23)} + \dots + C$$

14. Find the first four terms of the Taylor series for $f(x)=\frac{1}{x^2}$ at $a=1$.

$f(x) = x^{-2}$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\frac{1}{x^2}$	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24

$f(x) \approx 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3$

$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3$

15. (4pts each) If the statement is always true, write TRUE. Otherwise write FALSE and make the proper correction to make the statement true.

$\Delta x = \frac{5-2}{6} = \frac{1}{2}$

False (a) $\int_2^5 \sqrt{x}\,dx \approx S_6 = \cancel{\frac{1}{3}}\ \frac{1}{6}\left[\sqrt{2}+4\sqrt{5/2}+2\sqrt{3}+4\sqrt{7/2}+2\sqrt{4}+4\sqrt{9/2}+\sqrt{5}\right]$
Using Simpson's approximation with $n=6$.

False (b) If a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ was found to have a radius of convergence, $R=\infty$, then the ratio (or root) test resulted in a limit equal to $\cancel{\infty}$. 0

False (c) If f is an even function that is continuous on $[-a,a]$, then $\int_{-a}^{a} f(x)\,dx = \cancel{0}.\ 2\int_0^a f(x)\,dx$

False (d) If $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n=0}^{\infty} b_n$ is ~~convergent~~ inconclusive.

True (e) The distance from $x=0$ to $x=\pi/4$ along the curve $f(x)=\ln(\cos x)$ is equal to the value of $\int_0^{\pi/4} \sec x\,dx$.