The College Of WILLIAM & MARY

Math 111 Spring 2012
Lab 9

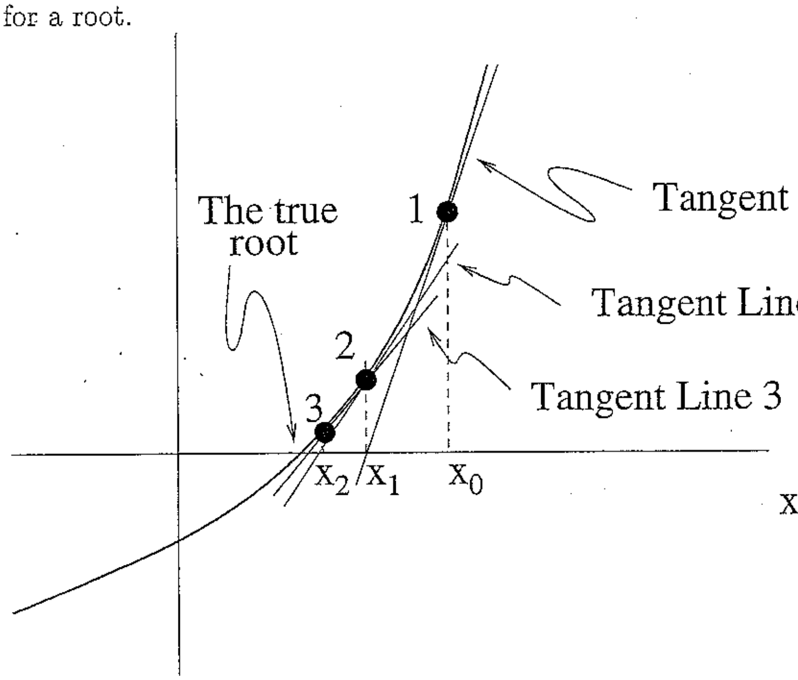
Newton's Method

In almost every application of mathematics, it is necessary to solve equations. When there is only one independent variable, the equation may be expressed in the form

$$f(x) = 0,$$

where $f(x)$ is usually a differentiable function of x. It might be a polynomial, a trigonometric function, an exponential or logarithm function, or a complicated combination of these. The equation may have one solution, or many solutions, and these x-intercepts are also called the roots, or the zeros of the function. Even when $f(x)$ is a polynomial, the only simple solution formula is the quadratic formula, which solves the equations for second degree polynomials. Functions other than polynomials rarely have formulas for their zeros.

We will study a method, called Newton's method, that uses derivatives to estimate the real zero of a differentiable function $f(x)$. Geometrically, this formula begins with an initial guess, x_0, for a root.



With extraordinary luck, x_0 would already be a root. That is, $f(x_0) = 0$. If not, we construct the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$. Then, because it is easy to find the x-intercept, x_1, of this straight line, we take that value as an improved guess.

One may actually determine whether, indeed, x_1 is an improved guess by comparing $f(x_0)$ with $f(x_1)$. It would be encouraging if the latter were closer to 0 than the former. Presuming that we are making progress (*i.e.*, $|f(x_1)| < |f(x_0)|$), we repeat the process to obtain a still better guess, x_2. The idea is to continue making progress (we hope) until we reach a guess x_n at which $f(x_n)$ is close enough to 0 to satisfy our needs. The sketch above shows the beginning of a successful search.

It remains only to write down some equations that will permit us to compute the successive guesses, x_n. The geometric picture makes this a fairly simple task.

Step 1: Select an initial guess for the x-intercept. Call this initial guess x_0. Test it by computing $f(x_0)$. If this is close enough to 0 (whatever "close enough" means), you can stop. Otherwise, go on to

Step 2: Compute the derivative, $f'(x_0)$, of $f(x)$ at x_0 to get the slope of the tangent line to the graph of $f(x)$ at x_0. Then write down the equation of this tangent line:

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

Step 2': Figure where this line crosses the x-axis by setting $y = 0$ in the previous formula. You get

$$x_1 = x_0 - f(x_0)/f'(x_0).$$

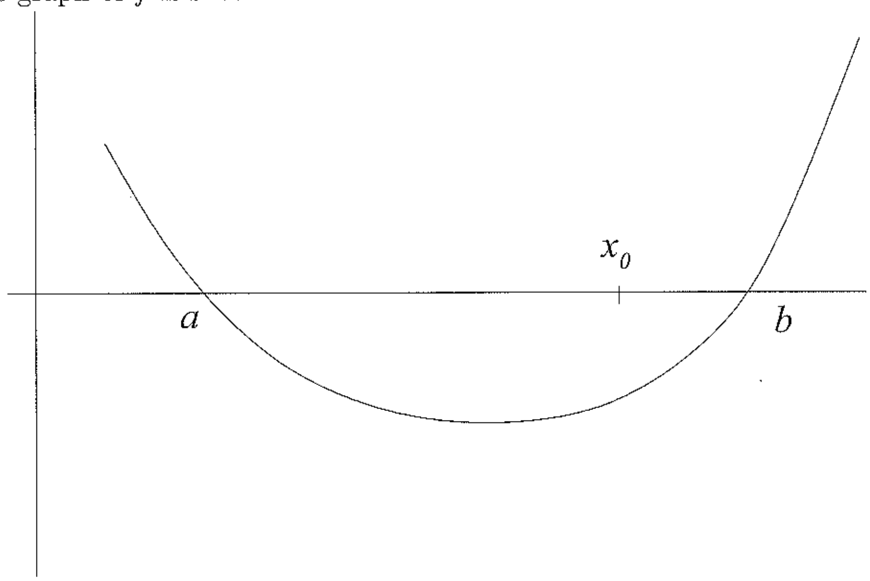
(If you like, you can memorize this formula for x_1.)

Step 3: Calculate $f(x_1)$ to see whether it is close enough to 0. If not, repeat step 2 and 2', replacing x_0 by x_1.

After multiple repetitions, Step 3 will read: Calculate $f(x_n)$ to see whether it is close enough to 0. If not, repeat step 2 and 2', replacing x_{n-1} by x_n.

If all goes well, this sequence of numbers will begin to converge to an x-intercept. That is, eventually, each new term of the sequence will better approximate the root than the previous term.

1. A Graphical Example.
Suppose the graph of f is shown below.



Note that there are two x-intercepts: one located at $x = a$, the other at $x = b$. We have marked the coordinate of an initial guess x_0. Now, apply Newton's Method.

a). Draw in the tangent line to the graph of f at $x = x_0$. Then, indicate where the x-intercept of this line is located and label it x_1. b). Continuing with Newton's Method, indicate the position of the next approximation, x_2. c). If we were to continue repeating Newton's Method to find $x_3, x_4, \ldots$, which x -intercept does the sequence appear to approach: $x = a$ or $x = b$?

2. A Computational Example.
Suppose $f(x) = x^3 - 2$. Enter this function as Y1 in your calculator. Our goal will be to apply Newton's Method to approximate an x-intercept. Let us choose the initial guess $x_0 = 1$. In all of the calculations that follow, use as many digits as your calculator will hold.

a). Find the equation of the tangent line to $f(x)$ at $x_0 = 1$. Then, find the x-intercept to his line and call it x_1. Show all calculations and work.

b). Repeat the procedure of step a) to find x_2. Show all calculations and work.

c). Find x_3. Show all calculations and work.

d). Based on your calculations of $x_1, x_2, \ldots$, write down an estimate for the numerical value of the x-intercept.

Instead of carrying out the procedure by hand as you did in problem 2, you can instead use the program Newton.

3. Using the Program Newton.

Your hand calculator will "automate" the Newton method. These are the instructions:

A. Enter the function f under Y1.

B. Hit PRGM and use the cursor keys to access the Newton program. Hit ENTER twice.

C. When the program Newton starts, make an initial guess x_0.

D. The program will generate a graph of f , the graph of the tangent line at $(x_0, f(x_0))$, and show the x-intercept of this tangent line. At the bottom of the screen, both x_1 and $f(x_1)$ appear.

E. To successively find : $x_2, x_3, \ldots$, continue to hit ENTER.

F. To quit the program, press ON and select QUIT.

Use Newton on your function $f(x) = x^3 - 2$. Tell how you decided to stop the iteration.

4. More Practice with the Program Newton.

Use the program Newton to estimate the numerical value of an x-intercept for the given function f and the initial guess x_0. In each case, apply the method three times. (That is, find x_1, x_2 and x_3.

a). $f(x) = x^3 - x - 5$ and $x_0 = 2$.

$$x_1 =$$

$$x_2 =$$

$$x_3 =$$

The numerical value of the x-intercept is approximately ____________.

b). $f(x) = 2x^4 - x - 1$ and $x_0 = -0.5$.

$$x_1 =$$
$$x_2 =$$
$$x_3 =$$

The numerical value of the x-intercept is approximately ____________.

Some Poor Examples.

In some cases, Newton's Method fails for one reason or another.

a). Consider the function $f(x) = \cos x$ and take the initial guess $x_0 = 0$. Without applying the Newton program, carefully explain why Newton's Method will fail to find a better x_1. (Hint: draw a graph!)

b). Consider the function $f(x) = e^x + 1$. Choose any x_0 you like. What happens to $x_1, x_2, \ldots$?
Do they converge to some x-intercept? Explain why or why not.

c). Suppose that f is a function having the following properties:

- $f(-1) = 2$ and $f(1) = 2$
- $f'(-1) = -1$ and $f'(1) = 1$.

If we pick $x_0 = 1$ as an initial guess, carefully explain why Newton's Method will fail to approximate an x-intercept in this case. (Hint: compute $x_1, x_2, x_3, \ldots$ and look for a pattern.)

5. Critical Points (the *other* critical points) for van der Waals.
One of principal reasons for finding zeroes of a function, $g(x)$ is that $g(x) = f'(x)$ for some function f. In other words, we are looking for the critical points of $f(x)$ so that we need to find the zeroes of its derivative.

Take the case of the van der Waals equation,

$$p = \frac{RT}{v-b} - \frac{a}{v^2}.$$

Using a good window, say, $.05 \leq v \leq .6$, one can display the graph of $p(v)$ for CO_2 at $T = 250°K$. That graph will give the user a crude idea of the location of the critical points for $p(v)$ between .05 and .6, for instance by using the TRACE button. In principal, these critical values for v should identify the extremes possible if CO_2 is either a supercooled vapor or a superheated liquid at 250°K.

Use the program NEWTON on your hand calculator to locate those zeroes more accurately. Use the original graph of $p(v)$ and data from the TRACE facility to make good guesses for starting values for NEWTON. For CO_2, the van der Waals parameters are $a = 3.592$ and $b = .04267$. The universal gas constant is $R = .0821$. (For units for these, see Lab 0).