Math 111 Spring 2012
Lab 10

The College Of WILLIAM&MARY

The Definite Integral as the Area under a Curve

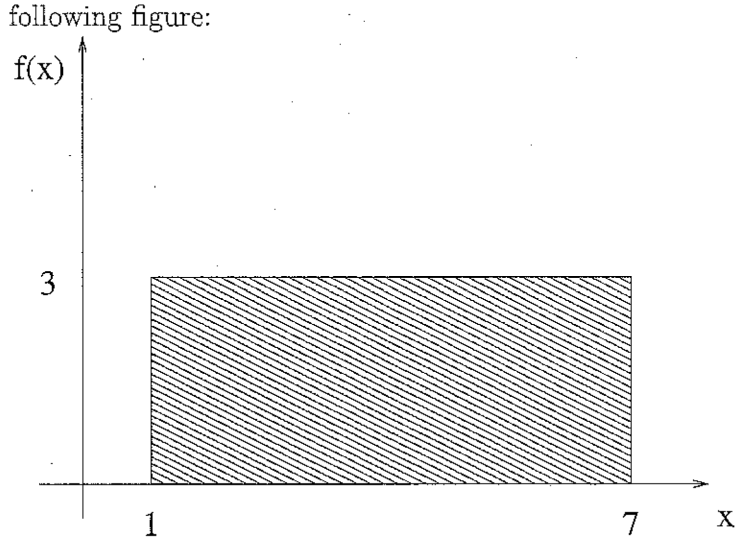
In calculus there are two major concepts. We are already familiar with one of them, the derivative. In this lab, we will introduce the other major idea, the definite integral. Just as there are several ways of understanding the derivative (i.e. as a rate of change, as the slope of a tangent line, etc.), there are many ways of understanding the definite integral. Here, we will focus on interpreting the definite integral as the area under a curve.

Definition: Suppose $f(x)$ is a function defined on the interval $[a, b]$ such that $f(x) \geq 0$. Hence, the graph of $f(x)$ lies above the x-axis. The symbol

$$\int_a^b f(x)dx$$

is then defined to be the area between the graph of f and the x-axis over the interval $[a, b]$. It is called the definite integral of f over the interval $[a, b]$. [1]

Example: Consider the constant function $f(x) = 3$. Lets evaluate (that is, find the numerical value of) the definite integral $\int_1^7 f(x)dx$. According to the above definition, the symbol $\int_1^7 f(x)dx$ may be interpreted as the area between the graph of $f(x) = 3$ and the x-axis, over the interval $1 \leq x \leq 7$. Since the graph of a constant function is a horizontal line, we have the following figure:



[1]This notion of "area under the curve" was accepted until the middle of the nineteenth century. The concept was superseded and refined by Riemann's definition. Riemann's version was, in turn, superseded by that of Henri Lebesgue. Lebesgue's definition is complicated. It is studied in later courses in mathematics.

Note that the region of interest is a rectangle. Its dimensions are 6 by 3 and so the area is 18. Therefore,

$$\int_1^7 f(x)dx = 18.$$

In this case, since $f(x) = 3$, we can also write

$$\int_1^7 3dx = 18.$$

1. A Constant Function. Consider the definite integral $\int_1^4 7dx$.
(a) What function is graphed for use as the "roof" of our area ? Show its graph below.

(b) What is the interval over which we are computing the area? Indicate this interval in the graph in part (a).

(c) Using simple geometry, evaluate (that is, find the numerical value of) $\int_1^4 7dx$.

2. A More General Result Find the general formula for $\int_a^b kdx$, where a, b, and k are constants (assume k is positive). Use simple geometry and explain your reasoning. (Hint: Draw a picture and label it completely.)

3. Consider the definite integral $\int_0^5 2x dx$.

(a) What function is graphed in this case ? Sketch the graph below.

(b) What is the interval over which we are computing the area? Indicate this interval in the graph in part (a).

(c) Use simple geometry to evaluate $\int_0^5 2x dx$.

4. A More General Formula. Find a general formula for $\int_0^b mx dx$, where m and b are positive constants. Use simple geometry and explain your reasoning.

5. Consider the definite integral $\int_{-3}^3 \sqrt{9 - x^2} dx$.

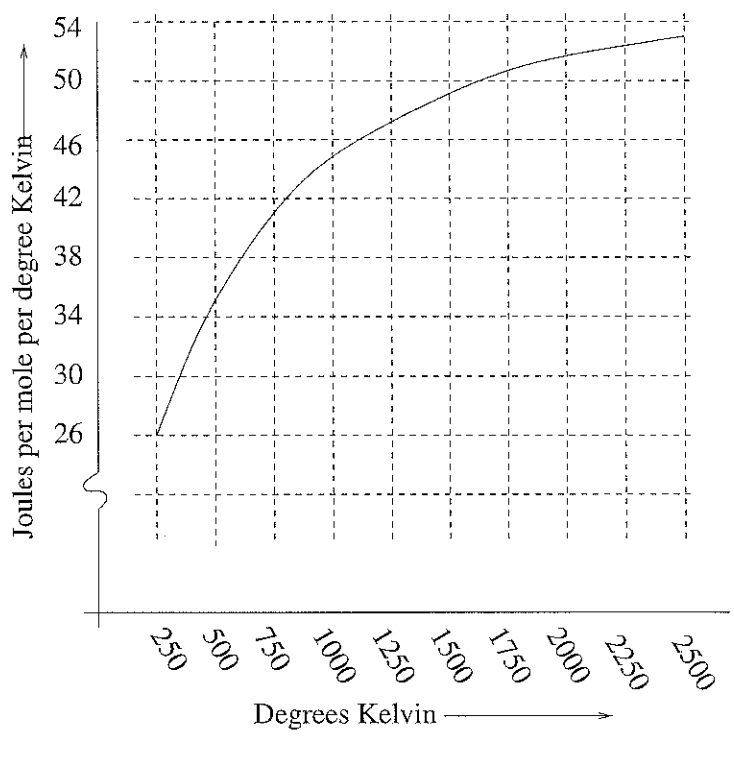
(a) What function is graphed in this case ? Graph it below. (In this case, you may use your graphing calculator.)

(b) According to your figure in part (a), the graph should appear to be the upper half of a circle. Show algebraically that $f(x) = \sqrt{9 - x^2}$ is the equation of a circle.

(c) Based on the discussion in parts (a) and (b), use simple geometry to evaluate $\int_{-3}^{3} \sqrt{9 - x^2}dx$. Explain your reasoning.

6. Numerical Estimates: Heating Carbon Dioxide

We decide to heat one mole of carbon dioxide gas from 500°K to 2250°K in a container of fixed volume. Below is a graph of $c_v(T)$, the molar heat capacity at constant volume for carbon dioxide.



Using the graph, estimate the amount of heat required to execute this task. Your answer should be in Joules. [Notice that the vertical axis of the graph has been shifted upward to save space.]