The College Of WILLIAM & MARY

Lab 1
Math 111 Spring 2012

Name: ______________________ Section: _____ Score: ______

New Functions from Old

In calculus, we tend to work with a collection of fairly standard functions, but we modify, transform, and combine these functions to fit a variety of problems and applications. In this first lab we will look at this collection of functions used in calculus and develop a toolbox of methods and techniques to work with these functions. As we progress through Math 111, you will want these tools at your fingertips, so watch carefully! (Note: you may use Module 1.3, "Transformations of Functions" on the CD that came with your textbook to help you with this material. It illustrates vertical and horizontal shifts and stretching and shrinking of functions.)

1. **Combining Functions**

One of the most basic ways we can modify functions is to combine them through simple mathematical operations such as addition, subtraction, division, and multiplication. The nice thing about this tool is that it generally gives predictable results that make sense. For example, consider the two functions $f(x) = 2\cos(x)$ and $g(x) = x$.

(a) Graph these functions on your calculator (enter $f(x)$ as Y_1 and $g(x)$ as Y_2). Make sure your calculator is in the radian mode and then use ZOOM-6:ZStandard to set the plot window to 10 in both x and y. Sketch the graph below.

(b) Now let's look at the combination $f(x) + g(x)$ which is also written $(f + g)(x)$. To do this, enter $Y_3 = Y_1 + Y_2$ into your calculator. You can access Y_1 and Y_2 in VARS > Y-VARS > Function. Graph this combination and then sketch the graph below. Write a statement describing why the graph makes sense for the combination of these two functions.

(c) Graph $(f - g)(x)$, that is $f(x) - g(x)$ and then sketch the graph below. Write a statement as to why this graph makes sense.

(d) Graph $f(x)g(x)$. Note: this is sometimes written $fg(x)$ which is NOT the same as $f(g(x))$. Sketch the graph below.

(e) Graph $\frac{f(x)}{g(x)}$ which is also written $\left(\frac{f}{g}\right)(x)$. Sketch the graph below. Write a statement describing any "problem" that occurs with this combination of functions and how it relates to the domain of the new function versus the domains of the original two functions.

2. **Translation: Shifts and Slides**

Visually, you can see how you could move functions around by just redrawing the "curve" in another location on the coordinate plane. This is called a translation. Algebraically, we can represent a translation by breaking the movement of the curve into two separate motions: one vertically in y and the other horizontally in x. For any $y = f(x)$, we can look at these translations as follows:

i) Vertical: The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up if $c > 0$ and down if $c < 0$.
ii) Horizontal: The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left if $c > 0$ and right if $c < 0$.

More generally, we say $y = a + f(x + b)$ is a function in which a and b control the vertical and horizontal translations, respectively, according to the rules above.

(a) When an object is thrown at some angle, it follows a parabolic trajectory. Consider an object thrown from the origin $(0,0)$ that has the trajectory $f(x) = 4x - 0.9x^2$. Use your calculator to sketch this function in the window $0 \le x \le 10$ and $0 \le y \le 10$. Sketch this below and label the trajectory, **A**.

(b) Now let's assume the object was thrown from $(0,4)$ rather than $(0,0)$. This would mean that the point of release was at a height of 4 rather than 0. Write the equation for the new trajectory below. Plot the new trajectory on your calculator and sketch the result on the graph above. Label this trajectory, **B**.

(c) Instead of shifting the graph up, assume the object was thrown from the point $(2,0)$ rather than $(0,0)$. This reflects a shift of 2 units to the right. Write the equation of this trajectory below. Plot this trajectory on your calculator and sketch the result on the graph above. Label this trajectory, **C**.

(d) Use the quadratic formula to solve the equation in part (c) to determine where the object lands.

3. **Scaling Functions**

Besides translations, we can transform functions by scaling them. As the name implies, scaling involves stretching or compressing a function and this can be done vertically and/or horizontally. For any function $y = f(x)$ and the "scale factor" $c > 0$,

i) The graph of $y = cf(x)$ stretches the graph of $y = f(x)$ vertically if $c > 1$ and compresses it if $c < 1$.

ii) The graph of $y = f(cx)$ compresses the graph of $y = f(x)$ horizontally if $c > 1$ and stretches it if $c < 1$.

4. **Reflecting Functions**

We can reflect a function about the *x*- or *y*-axis as follows:

i) The graph of $y = -f(x)$ reflects the graph of $y = f(x)$ about the *x*-axis.

ii) The graph of $y = f(-x)$ reflects the graph of $y = f(x)$ about the *y*-axis.

(a) Graph the function $f(x) = x^3 - 2x^2$ as Y_1 in your calculator. Sketch the curve below and label it **A**.

(b) Write the equation that stretches $f(x)$ horizontally by a factor of 3. Enter this into Y_2 and graph it. Sketch the result on the graph above and label it **B**.

(c) Take the function you generated in part (b), stretch it vertically by a factor of 4 and reflect it about the y-axis. Write the equation for this new function below. Enter the function into Y_3 and graph it. Sketch the result on the graph above and label it **C**.

5. **Modeling the behavior of a gas.**

We will use the reciprocal function, $p(v) = \frac{1}{v}$, where p is pressure and v is volume.

(a) Store this function as Y_1 and sketch the graph on your calculator.

(b) Chemists are only interested in this function for positive values of p and v. What are the most significant features of this portion of the graph?

One descriptive measure chemists use for the behavior of a gas is its equation of state. Given 1 mole of a gas, the molar ideal equation of state is

$$p(v) = \frac{RT}{v} \qquad (1)$$

Where $R = 0.08206$ liter-atmosphere per $^\circ K$, per mole is the universal gas constant.

Suppose we fix the temperature at $T = 273.2^\circ K$, which is about the temperature of freezing for water at 1 atmosphere of pressure, $0^\circ C$. So, $RT = 22.42$ liter-atmospheres per mole. (Notice how multiplication by T in degrees Kelvin removes that unit from the denominator of R).

(c) Use $RT = 22.42$ as a scaling factor to show the graph of $p(v) = RT \cdot \frac{1}{v}$. Store this function as Y_2 and graph it. Describe how these two functions compare with one another.

Now, we select the particular gas carbon dioxide, CO_2. For a single mole of carbon dioxide, the dependence of pressure on temperature and volume does not quite fit the ideal. One explanation is that from a microscopic point of view, the appearance of the volume in the equation of state is supposed to capture the amount of space in which an individual carbon dioxide molecule may roam. The Dutch physicist, Johannes van der Waals, suggested that the molecules of carbon dioxide themselves had to occupy some minimal volume. He, therefore, proposed that a better model for the available roaming space would subtract the minimal volume from the geometric volume. The parameter $b = 0.04267$ liters represents this minimal volume for a single mole of carbon dioxide.

(d) Store the function $p(v) = \frac{RT}{v-b}$, as Y_3 and graph it (fix the temperature $T = 250^\circ K$ [1]). You will need to re-calculate RT to get the correct scaling factor.

[1] For CO_2, this temperature is more interesting than $273.2^\circ K$.

(e) Using the viewing window $.05 \le v \le 10, 0 \le p \le 50$ graph the functions $p(v) = \frac{RT}{v}$ and $p(v) = \frac{RT}{v-b}$ from (c) and (d) above. Sketch the graphs in the space below.

Because the new equation of state still did not do a good job of representing the behavior of carbon dioxide, van der Waals went one step further in his microscopic analysis (and thereby won a Nobel prize) by making another correction. Thus, for one mole of carbon dioxide, he replaced the preceding version of p by subtracting $\frac{a}{v^2}$, where $a = 3.592\,\text{liter}^2$ - atmospheres. The complete van der Waal equation of state for one mole of carbon dioxide is[2]

$$p(v) = \frac{RT}{v-b} - \frac{a}{v^2}$$

(f) Using the numerical data for carbon dioxide, enter this function for $T = 250^\circ K$ as Y_4. Sketch the graph below. Use the window $.05 \le v \le .6, -14 \le p \le 50$.

(g) Describe the most significant features of this graph.

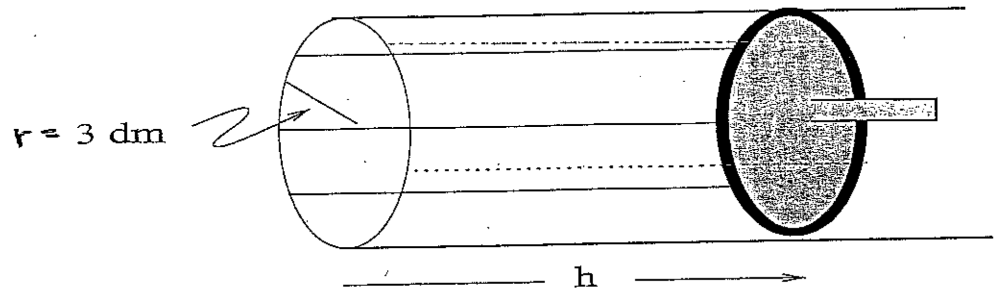
(h) Are there any unrealistic features of the van der Waals model? If so, describe these.

[2] Later, we will see a physical interpretation of the correction term, a/v^2

6. **Composite Functions**

One additional way to combine functions is by composition, where one function becomes the input to another. For example, if we had $f(x) = \sin(x)$ and $g(x) = 1 + x^2$, we can combine these functions through the composition $f(g(x)) = \sin(g(x)) = \sin(1 + x^2)$. The composition $f(g(x))$ is also written $(f \circ g)(x)$. A different composition would be $g(f(x)) = (g \circ f)(x) = 1 + \sin^2(x)$. Composite functions arise in any application where one function depends on another. As you will see in the next problem, composition can involve two or more functions.

One mole of an ideal gas $(p = RT/v)$ is contained in a circular cylinder that is equipped with a piston at one end. The cylinder has radius 3 *dm*. While the gas is held at the fixed temperature $T = 293.2^\circ K$, the piston is slowly pulled out of the cylinder at the rate of 2 *dm* per second.



If h is the distance from the inner surface of the piston to the base of the cylinder, then the volume of the cylinder is hA, where A is the area of the circular base. Initially $h = 10$ *dm*.

(a) Suppose the piston is moved outward at the rate of 2 *dm* per second. Write the volume of the cylinder as a function of t = time.

(b) Write the pressure, p, as a function of the volume v, and then as a function of time, t.

(c) How long does it take for the pressure to decrease to 50% of its original value?

(d) How far has the piston moved when this pressure is reached? Label the units in your answer.