The College Of
WILLIAM&MARY

Exploring the Derivative
(2.7, 2.8)

Prelab: Review Figure 1 (p. 141), Figure 6 (p. 143), Example 7 (p. 147) and Equation 2 (p. 152)

I. Introduction: We begin by exploring a tangent line geometrically. Suppose we have a function $y = f(x)$ as shown in the graph below.

Let P be a point on the given curve. Suppose we are interested in the tangent line to f at the point P.

Another line associated with the curve f is called a secant line. The secant line, PQ is the line that connects the points P and Q. Suppose the point Q moves along the curve and approaches P. The sketch also shows a few of the possible secant lines, PQ. Notice as Q approaches P, the slopes of the corresponding secant lines approach the slope of the tangent line at the point P.
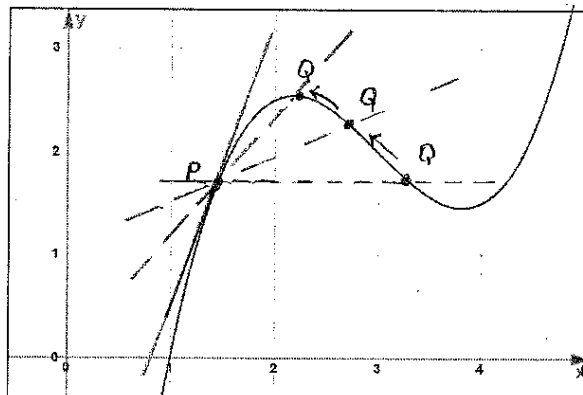
Notice as Q approaches P, we obtain the tangent line at the point P.

II. Definition of the Derivative (Geometric Version): Let f be a function and P be the point $(a, f(a))$. ***The derivative of f at the point P is defined to be the slope of the tangent line to the graph of f at the point P.*** The notation that denotes the derivative of f at the point where $x = a$ (that is $(a, f(a))$) is $f'(a)$.

III. Derivatives and Tables of Values: In many physical situations laboratory measurements produce data only at discrete points. In order to estimate the slope at one point from such data, we use the slope of a secant (or several secants).

Example 1: The Kelvin temperature of a fluid is an important indicator of its internal energy. In fact, part of the definition of an ideal gas is that its internal energy is completely determined by its temperature. While real gases may not obey the ideal equation of state, their energy is often assumed to depend only on temperature. In a wide temperature range, carbon dioxide is an example of such a gas. Along a particular adiabatic path, we have the following data for internal energy of carbon dioxide (u measured in Joules, J) as a function of temperature (measured in Kelvin, $^\circ K$).

Temperature ($^\circ K$)	u (J/mole)
290	6653
300	6939
310	7231

Table 1. Internal Energy of Carbon Dioxide as a Function of Temperature

(a) Calculate the "left-hand slope" of the secant that passes through the points at temperatures, $290\,^{\circ}K$ and $300\,^{\circ}K$ (b) Calculate the "right-hand slope" of the secant through points at temperatures $300\,^{\circ}K$ and $310\,^{\circ}K$. The tangent line to this internal energy curve at $T=300\,^{\circ}K$ has the exact slope denoted $\dfrac{du}{dT}$. (c) Use the two secant slope values to *estimate* the slope of this tangent.

IV. Derivatives and Formulas: In order to calculate a derivative of a function at a specific point we turn to algebraic formulas and limits. We begin with the slope of a secant line calculated as "rise over run" or "the difference in *y*-values over the difference in *x*-values" as taught in algebra. Select two points on the curve $y = f(x)$, one is the fixed point $(a, f(a))$ and the other is some non-fixed point, *x*, that is a small distance away from *a*. For these two points the "difference in *y*-values" is $f(x) - f(a)$ and the "difference in *x*-values" is $x - a$ so the slope of the secant line passing through $(a, f(a))$ and $(x, f(x))$ is
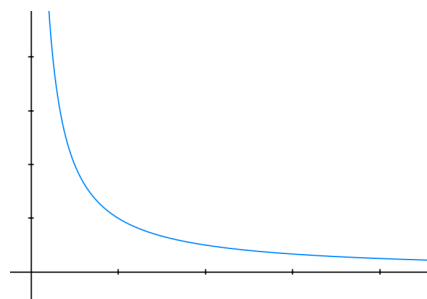
$$\frac{f(x) - f(a)}{x - a}$$

> Definition: **The derivative of a function *f* at a number *a*,** denoted $f'(a)$, is
> $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
> if this limit exists. This is Definition 5 in your textbook (p. 144).

Example 2: Use the definition above to find $f'(1)$ for the function $f(x) = \dfrac{1}{x}$. On the graph below, sketch the tangent line to *f* at $x = 1$. Does your value for $f'(1)$ make sense graphically?


The previous definition is particularly useful if one only cares to find the derivative of a function at a single value. Suppose we want to find the derivative, $f'(a)$, for *any* value of *x*, not just the single given $x = a$ value. Since we now consider *a* to be any value of *x*, we are interested in the derivative $f'(x)$. This will require some adjustments to our formula.

To determine the slope of any tangent line to the curve $y = f(x)$ at a point $(x, f(x))$, consider the secant line passing through this fixed point and a non-fixed point that is a small distance (h) away from x. This non-fixed point is located at $(x + h, f(x + h))$ so the slope of this secant line is

$$\frac{f(x+h) - f(x)}{h}$$

As this non-fixed point moves closer to x, h gets smaller and the secant line begins to take on the slope of the tangent line at $(x, f(x))$. This describes another limit.
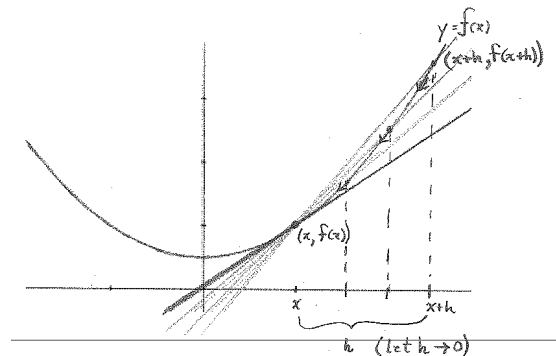
Definition: **The derivative of a function f is**
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
if this limit exists. This is Definition 2 in your textbook (p. 152).

Notice, calculating this limit does not generate a value but rather a new function of x, $f'(x)$.

Example 3: (a) Use the definition above to find $g'(x)$ for the function $g(x) = \dfrac{1}{\sqrt{x}}$.

Example 4: Let $h(P) = \dfrac{1}{\sqrt{P}}$. Find, (a) $h'(P)$

(b) $s'(P)$, where $s(P) = \dfrac{\sqrt{m}}{n+m} P - \dfrac{2m}{n\sqrt{P}}$; m and n are constants.

3-3

Solve the following problems ON THIS PAPER. You may use your textbook, lab, notes and peer collaboration (submit your own individual work, however). Do not use a calculator unless indicated.

1. (a) ***Use the limit definition on page 3-2*** of this lab to find $f'(2)$ where $f(x) = x - \dfrac{1}{x}$. Clearly show every step in evaluating this limit.

(b) Use your result in (a) to write the equation of the line tangent to $f(x)$ at $x = 2$.

2. (a) ***Use the limit definition on page 3-3*** of this lab to find $f'(x)$ where $f(x) = \sqrt{4-x}$. Clearly show every step in evaluating this limit.

(b) Find the domain of $f(x) = \sqrt{4-x}$.

(c) Find the domain of $f'(x)$, where $f(x) = \sqrt{4-x}$

(d) Use the proper limit definition (section 2.2) to find any vertical asymptotes of $f'(x)$.

(e) On the coordinate plane below, sketch the graph of $f(x) = \sqrt{4-x}$. Use this and your answers in 2(a), 2(c) and 2(d) to graph $f'(x)$. Clearly label each curve as well as intercept(s) and asymptote(s).