

Key Math 112 Fall 2001 Final Exam

1. (a) Show whether the following improper integral converges or diverges. If it converges, evaluate the integral.

$$\int_0^4 \frac{1}{x\sqrt{x}} dx$$

*Improper since
(undefined at $x=0$)*

$$= \lim_{a \rightarrow 0^+} \int_a^4 x^{-3/2} dx = \lim_{a \rightarrow 0^+} \left[-2x^{-1/2} \right]_a^4 = \lim_{a \rightarrow 0^+} \left(\frac{-2}{\sqrt{4}} + \frac{+2}{\sqrt{a}} \right)$$

$$\Rightarrow \infty$$

\therefore integral diverges

(b) Show whether this improper integral converges or diverges.

$$\int_1^{\infty} \frac{\cos^2 x}{\sqrt{x^3+1}} dx$$

Use a comparison test.

Compare to $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}} dx$ which

converges by The Comparison Theorem:

$$\frac{\cos^2 x}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}}$$



$$\int_1^{\infty} \frac{1}{x^{3/2}} dx$$

is a convergent p -integral since $p > 1$.

2. Evaluate each integral.

a)

Integration by parts

let $u = x$ $v = -\frac{1}{2} \cos 2x$
 $du = dx$ $dv = \sin 2x dx$

$$\begin{aligned} \int x \sin 2x dx &= -\frac{x}{2} \cos 2x - \int -\frac{1}{2} \cos 2x dx \\ &= -\frac{x}{2} \cos 2x + \frac{1}{2} \int \cos 2x dx \\ &= -\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x + C \end{aligned}$$

b)

u-substitution
 set $u = 1 + 2e^x$
 $du = 2e^x dx$

Rewrite 2

$$\int \frac{2}{e^{-x} + 2} dx = \int \frac{2}{\frac{1}{e^x} + \frac{2e^x}{e^x}} dx = \int \frac{2e^x}{1 + 2e^x} dx$$

$$\begin{aligned} &= \int \frac{du}{u} = \ln |u| + C \\ &= \ln(1 + 2e^x) + C \end{aligned}$$

c) use

Partial fractions

$$\int \frac{dx}{x^3 - x^2} = \int \frac{dx}{x^2(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \right) dx$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

$$1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$1 = (A+C)x^2 + (B-A)x - B$$

$$\left. \begin{aligned} A+C &= 0 \\ B-A &= 0 \\ -B &= 1 \end{aligned} \right\}$$

$$\left. \begin{aligned} B &= -1 \\ A &= -1 \\ C &= 1 \end{aligned} \right\}$$

So

$$\int \frac{dx}{x^3 - x^2} =$$

$$\int \left(\frac{-1}{x} + \frac{-1}{x^2} + \frac{1}{x-1} \right) dx$$

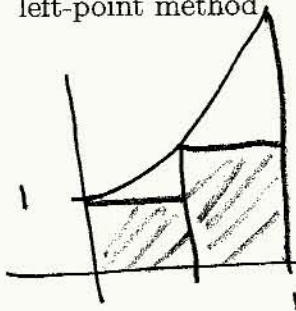
$$= -\ln|x| + \frac{1}{x} + \ln|x-1| + C$$

3. For each of the following methods, clearly explain, without doing numerical calculations, whether it gives an overestimate or underestimate of the integral

$$\int_0^1 e^{x^3} dx$$

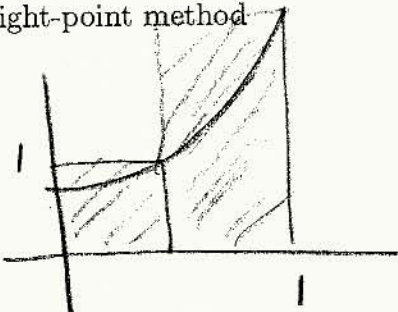
note: graph is increasing + concave up.

a) left-point method



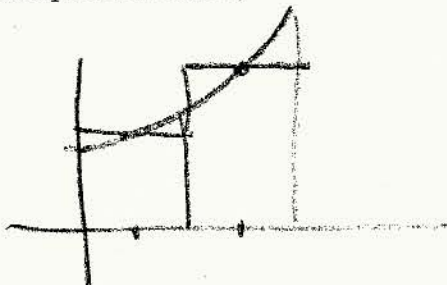
left rectangles are inscribed and are under the graph; \therefore underestimate the area

b) right-point method



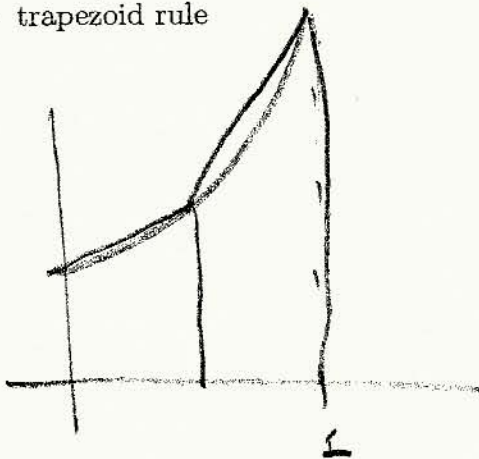
right rectangles \rightarrow over estimate

c) midpoint method,



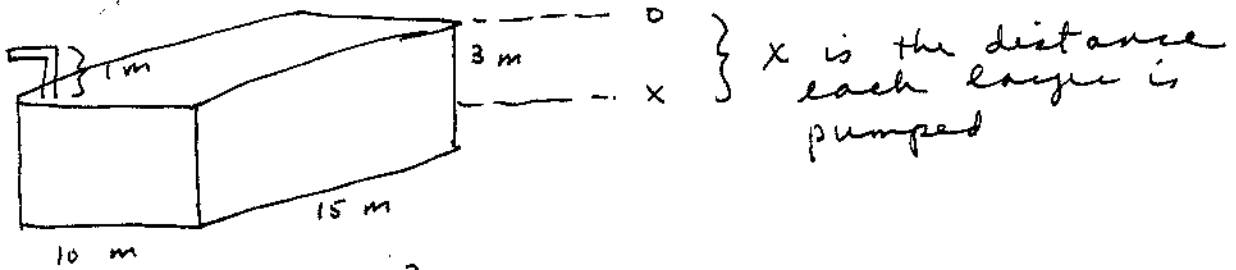
(do part (d) first)
midpoint rectangles give result that is opposite of trapezoid result. Since part (d) is an over estimate, part (c) is under.

d) trapezoid rule



Since curve is concave up, area of trapezoids is over

4. (a). Evaluate the work needed to pump the water out of a rectangular basement of the dimensions: depth 3 m, width 10 m, length 15 m. The basement is full of water, the pump is 1 m above the basement. (Note: The mass of water is 1000 kg/m^3 and $g = 9.8 \text{ m/s}^2$.)



$$\begin{aligned}
 W &= (1000)(9.8) \int_0^3 150(x+1) dx \\
 &= (9800)(150) \left(\frac{x^2}{2} + x \right) \Big|_0^3 = 11,025,000 \text{ J}
 \end{aligned}$$

(b) Find the hydrostatic force on one of the 10 m by 3 m walls of the basement from part (a) when the basement is full of water.

$$\begin{aligned}
 F &= (1000)(9.8) \int_0^3 10 \overset{\text{depth}}{\downarrow} x dx \\
 &= 441,000 \text{ N}
 \end{aligned}$$

5. Consider the area bounded by the following curves that are graphed below: $x + 4y - y^2 = 4$ and $y = x$.

$$x = 4 - 4y + y^2 = (2 - y)^2$$

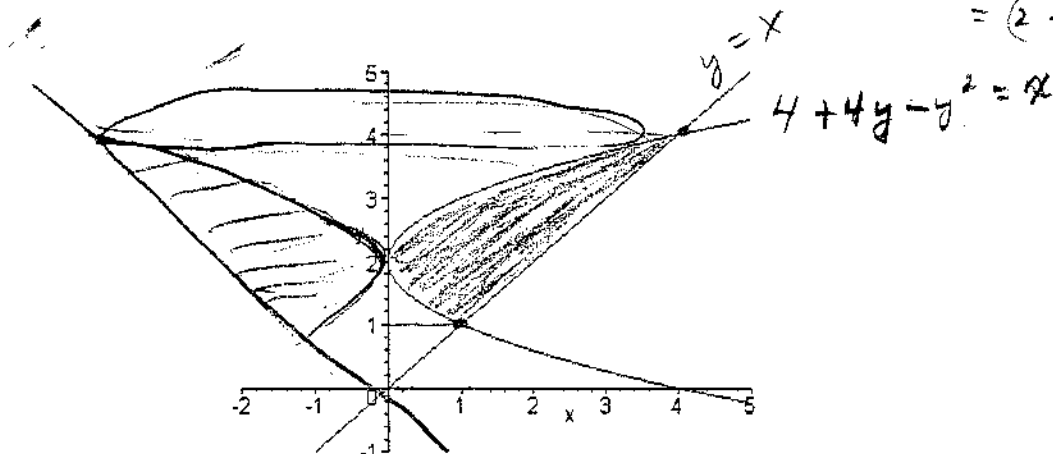


Figure 1: Graphs of $x + 4y - y^2 = 4$, $y = x$.

(a) Find the area of the region enclosed by the two curves.

$$\begin{aligned} \int_1^4 [y - (4 - 4y + y^2)] dy &= \int_1^4 (-y^2 + 5y - 4) dy \\ &= \left[-\frac{y^3}{3} + \frac{5y^2}{2} - 4y \right]_1^4 = \left(\frac{64}{3} + \frac{80}{2} - 16 \right) - \left(\frac{1}{3} + \frac{5}{2} - 4 \right) \\ &= \frac{9}{2} \end{aligned}$$

(b) Evaluate the volume of the solid obtained by rotation of the enclosed area of part (a) about the y -axis.

$$\begin{aligned} \text{Washers} \quad V &= \pi \int_1^4 [y^2 - (4 - 4y + y^2)^2] dy \\ &= \pi \int_1^4 (y^2 - (16 - 32y + 24y^2 - 8y^3 + y^4)) dy \\ &= \pi \int_1^4 (-16 + 32y - 23y^2 + 8y^3 - y^4) dy \\ &= \frac{72\pi}{5} = 14.4\pi \\ &\approx 45.2 \end{aligned}$$

6. (a) Compute the exact length of the curve given parametrically by $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ for $0 \leq t \leq 2$. (note x' is pos. so no reversals in direction.)

$$L = \int_0^2 \sqrt{(t^2)^2 + t^2} dt$$

$$= \int_0^2 \sqrt{t^4 + t^2} dt \quad \left(\begin{array}{l} \text{let } u = t^2 + 1 \\ du = 2t dt \end{array} \right)$$

$$= \int_0^2 t \sqrt{t^2 + 1} dt$$

$$= \frac{1}{2} \int_1^5 \sqrt{u} du = \frac{1}{2} \left[2 \cdot \frac{u^{3/2}}{3} \right]_1^5 \approx 3.39$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = t^2 \\ \frac{dy}{dt} = t \end{array} \right.$$

(b) The volume, V , in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Set up the integral needed to approximate the average volume of air in the lungs during one 5-second cycle. (Do not evaluate the integral.)

$$V_{av} = \frac{1}{5} \int_0^5 (0.1729t + 0.1522t^2 - 0.0374t^3) dt$$

7. a) Solve the differential equation $\frac{dy}{dx} = 2x + 2xy$ given that $y(0) = 4$. (Your solution should be in the form $y = \dots$)

$$\frac{dy}{dx} = 2x(1+y)$$

$$\int \frac{1}{1+y} dy = 2 \int x dx$$

$$\ln|1+y| = \cancel{x} \frac{x^2}{x} + C$$

$$|1+y| = e^{x^2} \cdot e^C$$

$$y+1 = A e^{x^2} \quad \text{where } A = \pm e^C$$

now plug in $x=0, y=4$ to solve for A and get $A=5$

$$\therefore y = 5e^{x^2} - 1$$

b) A tank contains 500 liters of brine with 10 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minute. The well-mixed solution drains from the tank at a rate of 8 liters per minute. How much salt is in the tank after 10 minutes?

Let y = amt. of salt in tank at time t .

$$\frac{dy}{dt} = \begin{matrix} \text{conc.} \\ (0) \end{matrix} \cdot \begin{matrix} \text{flow} \\ \text{rate} \\ (10 \text{ l/min}) \end{matrix} - \begin{matrix} \text{conc.} \\ \left(\frac{y}{500+2t} \right) \end{matrix} \cdot \begin{matrix} \text{flow} \\ \text{rate} \\ (8 \text{ l/min}) \end{matrix}$$

$$\frac{dy}{dt} = \frac{-8y}{500+2t}$$

$$\int \frac{1}{y} dy = -8 \int \frac{dt}{500+2t}$$

$$\ln|y| = \left(\frac{-8}{2} \right) \ln \left| \frac{250+t}{500+2t} \right| + C$$

$$y = A (250+t)^{-4} \quad \text{where } A = \pm e^C$$

let $t=0$ and $y=10$ to solve for $A \Rightarrow A = 3.9 \times 10^{10}$
(since there are 10 kg of salt in tank at time $t=0$)

$$y = 3.9 \times 10^{10} (250+t)^{-4}$$

$$y(10) = 3.9 \times 10^{10} (260)^{-4} \approx 8.55 \text{ kg salt}$$

note: net gain in volume of tank is 2 l/min (in 10, out 8 nets 2)

8. Consider the differential equation $\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{800}\right)$ as a model for a fish population in Lake Wobegon, where t is measured in weeks. (There are 200 fish in the lake at time $t=0$.)

a) Find the solution for the differential equation.

$$P(t) = \frac{800}{1 + A e^{-0.06t}} = \frac{800}{1 + 3 e^{-0.06t}}$$

where $A = \frac{800 - 200}{200} = \frac{600}{200} = 3$

$$\begin{aligned} \frac{dP}{dt} &= 0.06P\left(1 - \frac{P}{800}\right) \\ &= 0.06P - \frac{0.06P^2}{800} \\ &= 0.06P - 0.000075P^2 \end{aligned}$$

b) Find the number of fish in the lake after 21 weeks.

$$P(21) = \frac{800}{1 + 3 e^{-0.06(21)}} \approx 432$$

c) When is the population growing most rapidly? (This happens at the inflection pt. of the graph.)

set $P'' = 0$

$$P'' = 0.06 - 0.00015P = 0$$

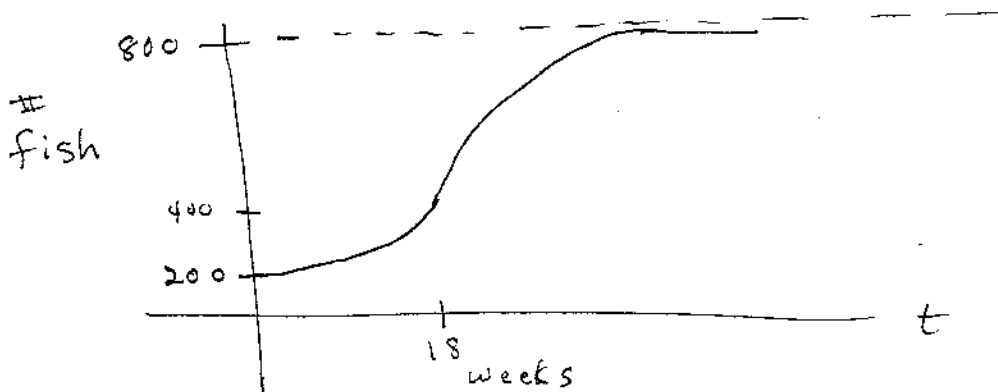
$$P = 400$$

Then solve for t using equation in part a) $\Rightarrow t \approx 18.3$ weeks

d) When will there be 300 fish in the lake?

$$300 = \frac{800}{1 + 3 e^{-0.06t}} \quad t \approx 10 \text{ weeks}$$

d) Sketch the graph of $P(t)$, where $0 \leq t \leq 100$.



9. (a) Calculate the sum of the infinite series

geometric series

$$a = \frac{3}{2}$$

$$r = \frac{3}{4}$$

$$\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} \frac{3}{2} \left(\frac{3}{4}\right)^{n-1} = \frac{3/2}{1-3/4} = \frac{3/2}{1/4} = 6$$

(b) Show whether the series converges or diverges. State the name of the test used.

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Limit Comparison Test

compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ (Known to diverge):

$$\lim_{n \rightarrow \infty} \frac{1/n}{1/(2n-1)} = \lim_{n \rightarrow \infty} \frac{2n-1}{n} = 2$$

\therefore both diverge

OR

Comparison Test

$$\text{since } \frac{1}{2n-1} > \frac{1}{2n}$$

and $\sum \frac{1}{2n}$ diverges ($\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$)

$\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also diverges.

OR Integral Test works too!

(c) Show whether the following series converges or diverges. If it converges, does it converge absolutely?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+4}$$

Converges by Alternating Series Test:

$$a) \lim_{n \rightarrow \infty} \frac{1}{n^3+4} = 0 \quad \checkmark$$

$$b) \frac{1}{(n+1)^3+4} < \frac{1}{n^3+4} \quad \checkmark$$

since $\sum_{n=1}^{\infty} \frac{1}{n^3+4}$ converges (use comparison test and compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (a convergent p-series))

the alternating series is absolutely convergent.

10. Consider the following power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$$

a) Find the interval of convergence of $f(x)$.

Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(x-3)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x-3}{5} \right| = \frac{|x-3|}{5}$$

when is this limit < 1 ?

$$\frac{|x-3|}{5} < 1 \Rightarrow |x-3| < 5 \Rightarrow -2 < x < 8$$

check endpoints:

if $x = -2$ series diverges

$x = 8$ series diverges

\therefore interval of convergence is $-2 < x < 8$

b) Find the radius of convergence of $f(x)$.

$$R = 5$$

c) Find the interval of convergence of $f'(x)$.

i) $f'(x)$ is convergent for $(-2, 8)$

ii) check endpoints

$$f'(x) = \sum_{n=1}^{\infty} \frac{n(x-3)^{n-1}}{5^n}$$

$$x = 8 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{5} \quad (\text{diverges as } a_n \not\rightarrow 0)$$

$$x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5} \quad (\text{diverges as } a_n \not\rightarrow 0)$$

$$\boxed{-2 < x < 8}$$

11. (a) Write the Maclaurin series for $f(x) = e^{-x}$.

$$\text{Known: } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substituting $-x$ for x :

$$\begin{aligned} f(x) = e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \end{aligned}$$

b) Find at least the first three terms of the Maclaurin series for $f(x) = x^2 e^{-x} + x$.

multiply the result from part (a) by x^2 and add x :

$$f(x) = \left[\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+2}}{n!} \right] + x$$

$$= x + x^2 \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right]$$

$$= x + x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!} - \dots$$