Lab 4: Applications of integration: work

Use of the definite integral

In previous work, we have studied several ways of evaluating integrals. With these tools in hand, it is possible to explore some of the many applications of the definite integral by solving problems in areas such as geometry and physics. There are many situations in which the quantity of interest may be expressed as a definite integral.

The general procedure, which we followed when we found the area under curves, is

(a) Chop the desired quantity into very thin slices.

(b) Within each slice, calculate an approximation to the desired quantity.

(c) Add up the results of all of the slice approximations from Step 2. The resulting sum is a Riemann sum that approximates the desired total quantity.

(d) Obtain a definite integral by taking the limit of the Riemann sum in Step 3 as the slices get thinner and thinner.

For instance, in computing the area under a curve, the interval of interest was divided into n smaller subintervals. Summing the areas under the curve over each subinterval yielded a Riemann sum, and the definite integral came about as the limit of the Riemann sum as the subintervals became smaller and smaller.

In many problems however, the difficulty lies in recognizing the quantity that we want as a Riemann sum. Therefore, the key in using the technique of "slicing and summing" is to properly divide the quantity of interest into small subdivisions, the sum of which leads to a Riemann sum. Then, taking the limit as the subdivisions get smaller and smaller leads to a definite integral.

4.1 Integration in Matlab

You can perform integration of symbolic quantities in Matlab using the `int`. Before you start the lab, take a look at how Matlab evaluates the indefinite and definite integrals.

4.1.1 Indefinite integrals

To evaluate the indefinite integral $\int x^2\, dx$, enter the following:

```
>> int(x^2)     % Don't forget to declare x a symbol with the syms command!
```

If your integrand contains has more than one symbolic quantity, it's a good idea to declare the variable of integration explicitly:

```
>> syms a, x
>> int(a*x, x)    % This is the integral of a*x with respect to x,
>> int(a*x, a)    % while this is the integral of a*x with respect to a!
```

4.1.2 Definite integrals

Definite integrals are also computed using the `int` command, but a range of integration must be given. For instance, to compute $\int_1^5 x^2\,dx$, we use

```
>> int(x^2, 1, 5)
```

The material in this lab is a set of applications of integration. The hard part will be setting up the appropriate integrals. You may use MATLAB to evaluate the integrals using the commands you just learned.

4.2 Applications of integration

4.2.1 Force and work

When a force acts on some body (such as a weight bearing on surface) and that body is moved a certain distance in the same direction as the force, the **work** done by the force is defined to be

$$\text{work} = \text{force} \ \times \ \text{distance}.$$

In many situations, the force varies with the distance displaced so that it is natural to cut the displacement into small pieces and multiply each small displacement by an estimate of the acting force:

$$\text{force} \ \times \Delta s,$$

where s measures displacement *in the direction of the force.*

Exercise 4.1 For example, suppose water is to be raised from a well 40 ft. deep by means of a bucket attached to a rope. In the problems below, the force of interest is the tension in the rope, perhaps as exerted by someone at the well-head. When the bucket is full, it weighs 30 lbs. Assume that that the tension in the rope exactly matches the weight in the bucket.*

(a) Neglecting the weight of the rope, find the work done by the tension on the rope as the bucket rises to the top of the well.

(b) Now, suppose the bucket has a leak that causes it to lose water at the constant rate of 0.25 lb. for each foot the bucket is raised. If height is measured from the water level in the well, what is the weight of the bucket at x ft above the water?†

*In reality, the tension in the rope is larger than the weight of the bucket.

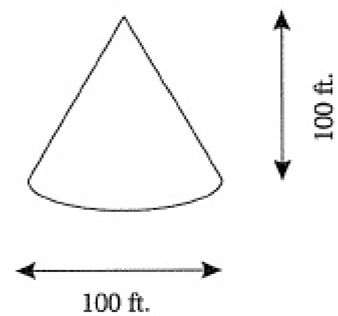
†This is artificial—leakage should be a *time* dependent, not *displacement* dependent phenomenon. The problem is stated this way so as to fit into the structure of tools available to us right now. The mathematics is driving the model instead of the model driving the mathematics.

(c) Using the result from part b, what is the work done by the tension in the rope as the bucket moves a short distance Δx? [Remember, the force, that is, the tension in the rope, depends on the current position, x, of the bucket.]

(d) Using the results from part (c) write the integral needed to find the total amount of work done by the tension in the rope as the leaking bucket in raised to the top of the well. You may evaluate the integral by hand or using MATLAB.

4.2.2 Some conestruction problems

Exercise 4.2 A monument in the shape of a cone is built to a height of 100 ft. The base has a diameter of 100 ft. and the bricks used in construction weigh 2 lbs/ft^3.

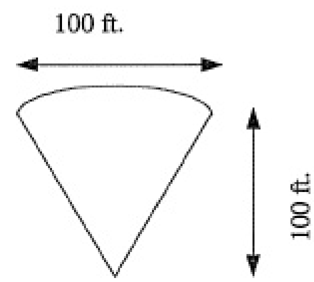


(a) How should this monument be sliced to ultimately calculate the work required to build it—vertically or horizontally? Sketch your slicing scheme below. (Hint: work is being performed to counteract the force of gravity.)

(b) In finding the work required to construct the ith slice of the monument, we need to know the weight of the bricks in that slice. We know that the bricks weigh 2 lb/ft^3, so multiplying by the volume of the slice will give the weight of the bricks in the slice. If height, h, is measured from the base of the monument, find the volume of the ith slice of the monument. (Hint: Use similar triangles to find the radius of the slice as a function of h.)

(c) Using the result of part (b), find and evaluate a definite integral that represents the total work required to build the monument. You may show your work by hand or attach your MATLAB worksheet.

Exercise 4.3 Suppose that the same conical structure of the preceding exercise is to be built in another location, except that it will be inverted:



(a) Neglecting any additional structure to keep the monument inverted, write a definite integral that represents the total work required to build the monument. Label any variables you use on the diagram. Use MATLAB to evaluate the integral. Attach your MATLAB worksheet to this page.

(b) Considering the definition of work given previously, provide an intuitive explanation of the results in this and the preceding problem. In other words, explain why it takes more work to build the monument upside down.