S'08

1) Compute $\int x \sin x \, dx$, showing all your steps.

I by Parts :

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

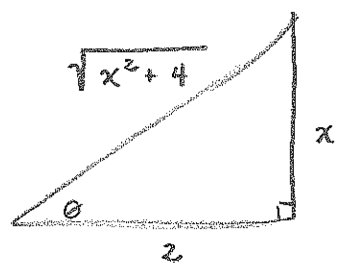
$$= -x\cos x + \sin x + C$$

2) Compute $\int \frac{dx}{(x^2+4)^{3/2}}$.

Trig Sub :

let $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$



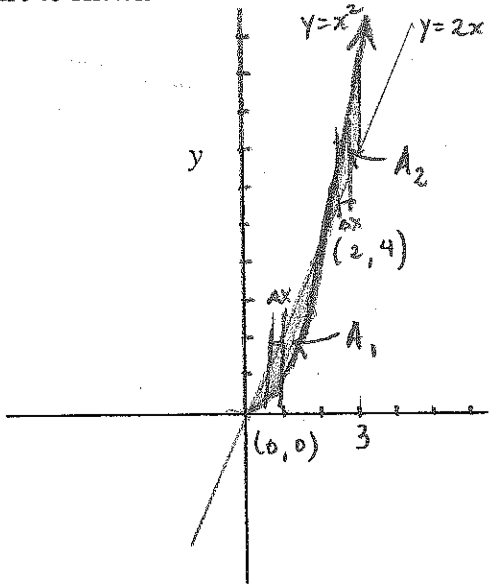
$$= \int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^{3/2}} \, d\theta$$

$$= \int \frac{2\sec^2\theta}{\left(2\sqrt{\tan^2\theta+1}\right)^3} \, d\theta$$

$$= \frac{2}{8}\int \frac{\sec^2\theta}{(\sec\theta)^3} \, d\theta = \frac{1}{4}\int \frac{1}{\sec\theta} \, d\theta = \frac{1}{4}\int \cos\theta \, d\theta$$

$$= \frac{1}{4}\sin\theta + C = \frac{1}{4}\left[\frac{x}{\sqrt{x^2+4}}\right] + C = \frac{x}{4\sqrt{x^2+4}} + C$$

3) Compute the area bounded by the curve $y = x^2$ and the line $y = 2x$ for $0 \le x \le 3$. As your first step, make sure to sketch and shade the area to be computed.



Intersection(s):

$$x^2 = 2x$$

$$x(x-2) = 0 \therefore x=0,\ x=2$$

$$A_1 = \int_0^2 2x - x^2\,dx$$

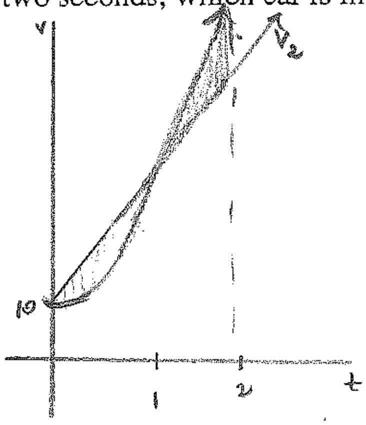
$$A_2 = \int_2^3 x^2 - 2x\,dx$$

$$A_{total} = A_1 + A_2 = \int_0^2 2x - x^2\,dx + \int_2^3 x^2 - 2x\,dx$$

$$= x^2 - \frac{x^3}{3}\Big|_0^2 + \frac{x^3}{3} - x^2\Big|_2^3$$

$$= 4 - 8/3 + \left[27/3 - 9 - (8/3 - 4)\right] = 8/3$$

4) Two cars cross the starting line at the same moment. The velocity of the first is given by $v_1(t) = 10 + t^2$ and the velocity of the second is $v_2(t) = 10 + t$, with time measured in seconds and velocity in feet per second. After two seconds, which car is in the lead, and by how much?



$$v_1(2) = 10 + 4 = 14\ \text{ft/s}$$

$$v_2(2) = 10 + 2 = 12\ \text{ft/s}$$

$v_1(t) = v_2(t)$ when $10 + t^2 = t + 10$

$$t(t-1) = 0 \quad t = 0,\ t = 1$$

$$d_1 = \int_0^2 v_1(t)\,dt = \int_0^2 10 + t^2\,dt = 10t + \frac{t^3}{3}\Big|_0^2$$

$$= 20 + 8/3 = 68/3\ \text{ft}$$

$$d_2 = \int_0^2 v_2(t)\,dt = \int_0^2 10 + t\,dt = 10t + \frac{t^2}{2}\Big|_0^2$$

$$= 20 + 2 = 22\ \text{ft}$$

$$= 66/3\ \text{ft}$$

Car 1 is in the lead by $68/3 - 66/3 = 2/3$ ft

5) Solve the differential equation $\frac{dP}{dt} = \frac{P}{t^2}$, given $P(1) = e$.

$$\frac{dP}{dt} = \frac{t^{-2}}{P^{-1}}$$

$$\int \frac{1}{P}\,dP = \int \frac{1}{t^2}\,dt$$

$$\ln|P| = -\frac{1}{t} + C \qquad P(1) = e \therefore \ \ln e = -1 + C$$

$$1 = -1 + C$$

$$C = 2$$

$$\ln|P| = -\frac{1}{t} + 2$$

$$e^{-1/t + 2} = P$$

$$P(t) = e^2 e^{-1/t}$$

6) If $\int_1^3 f(u)\,du = 7$, find the value of $\int_1^2 f(5-2x)\,dx$.

let: $u = 5 - 2x$

$du = -2\,dx$

$-\frac{1}{2}du = dx$

when: $x = 1,\ u = 5 - 2 = 3$

$x = 2,\ u = 5 - 4 = 1$

$$\int_1^2 f(5-2x)\,dx = -\frac{1}{2}\int_3^1 f(u)\,du$$

$$= \frac{1}{2}\int_1^3 f(u)\,du$$

$$= \frac{1}{2}[7] = \frac{7}{2}$$

7) Economists have noticed that a single expenditure of size $D can have an economic impact larger than $D on an economy because the people who receive the payment will re-spend part of it. For example, historical studies might show that members of a community tend to spend 80 percent of whatever they receive. Consequently, the recipients of an initial expenditure of $100 will spend $80, and the recipients of that $80 will in turn spend $64, and so on forever. Explain why economists say that the $100 initial expenditure in the example above will have a total economic impact of $500.

Let E_n be expediture w/ $n=1$ for initial

$E_1 = 100$

$E_2 = .8(100)$

$E_3 = .8[.8(100)] = (.8)^2(100)$

$E_4 = .8[.8(.8(100))] = (.8)^3(100)$

$\vdots$

$E_n = (.8)^{n-1}(100)$

$\therefore E_{Total} = \sum_{n=1}^{\infty} (.8)^{n-1}(100)$ is geometric w/ $a=100$, $r = 4/5 < 1$ $\therefore$ converges

$\text{Sum} = S = \dfrac{a}{1-r} = \dfrac{100}{1-4/5} = \$500$

8) For which values of x does the infinite series $\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n x^n$ converge? Circle your answer.

power series

a) only at $x = 0$.

b) for all values of x.

c) only for the x values $-\frac{3}{2} < x < \frac{3}{2}$.

d) only for the x values $-1 < x < 1$.

e) none of the above.

Ratio Test: $\lim_{n\to\infty}\left|\dfrac{(2/3)^{n+1}x^{n+1}}{(2/3)^n x^n}\right| = \lim_{n\to\infty}|x|\cdot 2/3$

$= 2/3|x| < 1$

$|x| < 3/2$

$-3/2 < x < 3/2$

Test Endpoints

for $x = -3/2$: $\sum_{n=1}^{\infty}(-2/3)^n(-3/2)^n = \sum_{n=1}^{\infty} 1^n$ div. (T4D $\lim_{n\to\infty} 1^n = 1 \neq 0$)

$x = 3/2$: $\sum_{n=1}^{\infty}(-2/3)^n(3/2)^n = \sum_{n=1}^{\infty}(-1)^n$ div (Alt series test $\lim_{n\to\infty} 1^n = 1 \neq 0$)

9) If the statement is always true, write TRUE. Otherwise, write FALSE. No explanation is necessary.

Just FYI for key

TRUE a) $\int_1^{\infty}\frac{1}{x^{\sqrt{2}}}dx$ is convergent. $p = \sqrt{2} > 1$

TRUE b) $\frac{x^2+4}{x^2(x-4)}$ can be put in the form $\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-4}$, where A, B, and C are constants. repeated linear factor

FALSE c) The equation $y' = x + y$ is separable.

FALSE d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. (This is inconclusive)

TRUE e) If $a_n > 0$ and $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges. Implies absolute convergence

FALSE f) The Ratio Test can be used to determine whether $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges.

$\lim_{n\to\infty}\left|\dfrac{1}{(n+1)^3}\cdot\dfrac{n^3}{1}\right| = 1 \therefore$ inconclusive

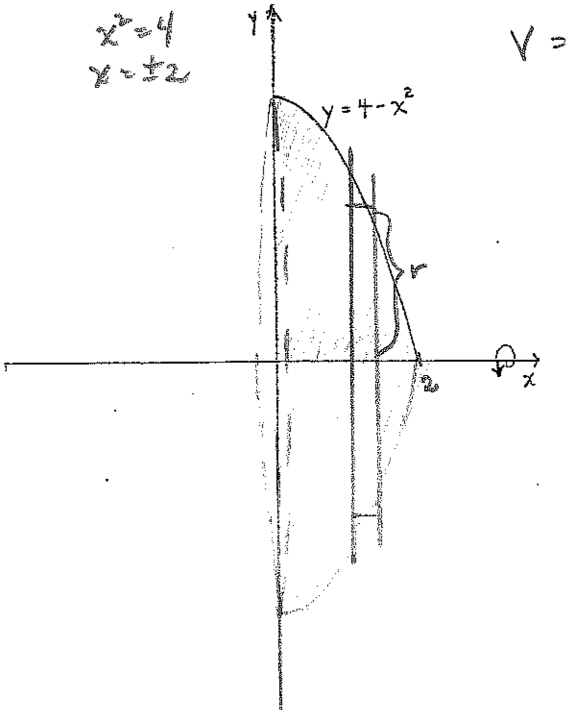
10) Suppose R is the region bound by the parabola $y = 4 - x^2$, $y = 0$, and $x = 0$ as shown in the figure below. Set up, and solve the integral needed to find the volume of the solid created if R is rotated about the x-axis.

$0 = 4 - x^2$

$x^2 = 4$

$x = \pm 2$

$A_{slice} = \pi r^2 = \pi(4 - x^2)^2 = \pi(16 - 8x^2 + x^4)$



$$V = \pi \int_0^2 16 - 8x^2 + x^4 \, dx$$

$$= \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2$$

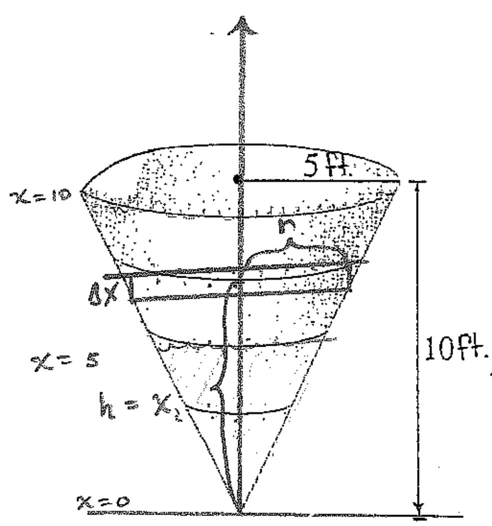
$$= \pi \left[16(2) - \frac{8}{3}(8) + \frac{32}{5} \right] = \frac{256\pi}{15}$$

-or-

$= 17.067\pi$

11) A water tank is in the shape of a right circular cone of altitude 10 feet and base radius 5 feet, with its vertex on the ground as shown in the figure below. If the tank is full, find the work done in pumping enough water out of the top of the tank, to leave the water level at 5 feet. The weight density of water is $62.5 \frac{lb}{ft^3}$.

You must indicate the location of zero on your sketch.



$W = F \times \text{distance}\,(d) = V \cdot \delta \cdot d$

Volume of a slice, $V = \pi r^2 \Delta x$

need r in terms of x (use similar triangles)

$\frac{h}{r} = \frac{x}{r} = \frac{10}{5}$

$r = \frac{1}{2}x$

$V = \pi\left(\frac{1}{2}x\right)^2 \Delta x$

$= \frac{\pi}{4} x^2 \Delta x$

$F_i = V \cdot \delta = \frac{\pi}{4} x^2 \Delta x (62.5)$

distance any slice must travel, $d_i = 10 - x_i$

Work to move one slice: $W_i = F_i \cdot d_i = \frac{\pi}{4} x^2 \Delta x (62.5)(10 - x)$

$$W_{Total} = \frac{62.5}{4} \pi \int_5^{10} x^2 (10 - x)\, dx = \frac{62.5}{4} \pi \left[\frac{10}{3}x^3 - \frac{1}{4}x^4 \right]_5^{10} = 8951.823\pi \text{ ft-lb}$$

12) Evaluate the integral $\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$

$$\begin{array}{r} 2x \\ x^2-2x-8\,\overline{)\,2x^3-4x^2-15x+5} \\ -(2x^3-4x^2-16x) \\ \hline x+5 \end{array}$$

$$= \int 2x + \frac{x+5}{x^2-2x-8}\,dx$$

PFD: $\frac{x+5}{x^2-2x-8} = \frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

$x+5 = Ax+2A+Bx-4B$

$-2(A+B=1)$

$2A-4B=5$

$-6B=3$

$B=-\frac{1}{2}$

$A=\frac{3}{2}$

$$= \int 2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\,dx$$

$$= x^2 + \frac{3}{2}\int \frac{1}{x-4}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx$$

$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

13) Evaluate the integral $\int_0^{\pi/2} \sqrt{\sin x}\,\overbrace{\cos^3 x}^{\cos^2 x\cdot\cos x}\,dx$.

$$= \int_0^{\pi/2} (\sin x)^{1/2}(1-\sin^2 x)\cos x\,dx$$

let $u=\sin x$

$du = \cos x\,dx$

when $x=0$, $u=0$

$x=\frac{\pi}{2}$, $u=1$

$$= \int_0^1 u^{1/2}(1-u^2)\,du$$

$$= \int_0^1 u^{1/2} - u^{5/2}\,du = \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}\Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{7} = \frac{14-6}{21} = \frac{8}{21}$$

14) Evaluate the improper integral $\int_{-1}^{2} \frac{1}{x^2}\,dx$.   (discontinuity @ $x=0$)

$$\lim_{t\to 0^-} \int_{-1}^{t} x^{-2}\,dx + \lim_{t\to 0^+} \int_{t}^{2} x^{-2}\,dx$$

$$= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{2}$$

$$= \lim_{t\to 0^-} -\cancel{\frac{1}{t}}^{\,-\infty} - 1 = -\infty \;\therefore\; \text{divergent}$$

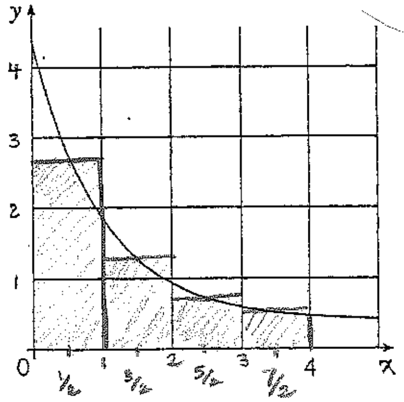
(DNE)

15) Approximate the integral $\int_0^4 f(x)\,dx$ using the curve below. Show the geometric interpretation of each approximation on the graph.   Approximate heights from sketch ∴ may vary some.

(a) Midpoint Rule, $n=4$

$$M_4 = \Delta x \cdot \sum_{i=1}^{4} f(\bar{x}_i)$$

$$\approx 1\,[2.8 + 1.3 + .7 + .5] = 5.3$$

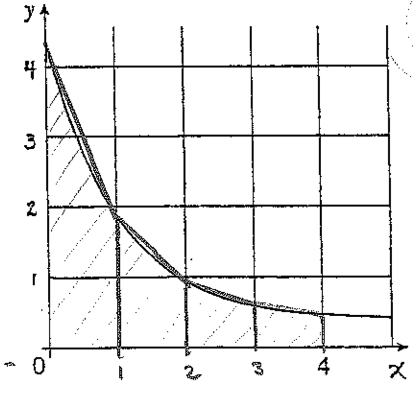


(b) Trapezoid Rule, $n=4$

$$T_4 = \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right]$$

$$\approx \frac{1}{2}\left[4.3 + 2(1.8) + 2(.9) + 2(.6) + .5\right]$$

$$= 5.7$$



16) Find the length of the curve, $y = \ln(\cos x)$, over the interval $0 \le x \le \frac{\pi}{3}$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$y = \ln(\cos x)$$

$$y' = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$[y']^2 = \tan^2 x.$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x \, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$

$\cos \pi/3 = 1/2$

$\therefore \sec \pi/3 = 2$

$\tan \pi/3 = \sqrt{3}$

$$= \ln|\sec \pi/3 + \tan \pi/3| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2 + \sqrt{3}| - \ln 1$$

$$= \ln(2 + \sqrt{3})$$

17) Determine whether each *sequence* below converges or diverges, using calculus methods taught in class. If the *sequence* converges, to what value does it converge?

a) $a_n = (-1)^n \dfrac{n+2}{n^2+4}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(n+2)/n^2}{(n^2+4)/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} + \cancel{2/n^2}^{\,0}}{1 + \cancel{4/n^2}_{\,0}} = 0$$

$\therefore$ a_n also converges to 0

b) $a_n = \dfrac{e^n}{3^n} = \left(\dfrac{e}{3}\right)^n$

since $\frac{e}{3} < 1$

$$\lim_{n\to\infty} \left(\frac{e}{3}\right)^n = 0$$

$\therefore$ sequence converges to 0

18) Determine whether the series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ converges or diverges. If it converges, find the sum.

PFD:

$\frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$

$4 = An + 2A + Bn$

$2A = 4$

$A = 2$

$A + B = 0$

$B = -2$

$= \sum_{n=1}^{\infty} \left[\frac{2}{n} - \frac{2}{n+2}\right]$ Telescoping Series

$S_n = (2 - \cancel{\tfrac{2}{3}}) + (1 - \cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{2}{3}} - \cancel{\tfrac{2}{5}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + \dots$

$\dots + (\cancel{\tfrac{2}{n-2}} - \cancel{\tfrac{2}{n}}) + (\cancel{\tfrac{2}{n-1}} - \tfrac{2}{n+1}) + (\cancel{\tfrac{2}{n}} - \tfrac{2}{n+2})$

$S_n = 2 + 1 - \frac{2}{n+1} - \frac{2}{n+2}$

$\lim_{n\to\infty} 3 - \cancel{\frac{2}{n+1}}^{0} - \cancel{\frac{2}{n+2}}^{0} = 3$

∴ Series converges, sum = 3

19) Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Integral test:

$f(x) = \frac{1}{x(\ln x)^2}$ is continuous and positive for $x \geq 2$

Since $\frac{1}{(n+1)[\ln(n+1)]^2} < \frac{1}{n(\ln n)^2}$, $a_{n+1} < a_n$ ∴ related function $f(x) = \frac{1}{x(\ln x)^2}$ is dec.

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ let $u = \ln x$, $du = \frac{1}{x}\,dx$ $\int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{\ln x}$

$= \lim_{t\to\infty} -\frac{1}{\ln x}\Big|_2^t = -\left[\lim_{t\to\infty} \cancel{\frac{1}{\ln t}}^{0} - \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$

∴ convergent

20) Determine whether the series $\sum_{n=1}^{\infty}\left|\frac{(-1)^n 10n}{n^2+1}\right|$ is absolutely convergent, conditionally convergent or divergent. ← test for abs. conv.

Comparison Test:

$$\frac{10n}{n^2+1} < \frac{10n}{n^2} = \frac{10}{n} \text{ and } \sum_{n=1}^{\infty} \frac{10}{n} \text{ is harmonic } \therefore \text{divergent (inconclusive)}$$

Limit Comparison Test:

$$\lim_{n\to\infty} \frac{10n}{n^2+1}\cdot\frac{n}{10} = \lim_{n\to\infty}\frac{10n^2}{10n^2+10} = 1 > 0 \therefore \text{ BOTH series diverge}$$

So, series is NOT absolutely convergent

Alternating Series Test:

i) $f(x) = \frac{10x}{x^2+1}$, $f'(x) = \frac{(x^2+1)(10) - 10x(2x)}{(x^2+1)^2} = \frac{10(1-x^2)}{(x^2+1)^2} < 0$

when $1-x^2 < 0$

$x^2 > 1$

$\therefore$ decreasing ✓

ii) $\lim_{n\to\infty} \frac{10n/n^2}{n^2/n^2 + 1/n^2} = \lim_{n\to\infty} \frac{\frac{10}{n}\to 0}{1+\frac{1}{n^2}\to 0} = 0$ ✓

Series is conditionally convergent

21) Determine whether the series $\sum_{n=1}^{\infty} e^{-n}\, n!$ is convergent or divergent.

$$= \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!\cdot e^n}{e^{n+1}\; n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,\cancel{e^n}}{\cancel{e^n}\cdot e\cdot \cancel{n!}} = \frac{1}{e}\lim_{n\to\infty} n+1 = \infty > 1$$

$\therefore$ divergent

22) Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n(n+1)}$.

Ratio Test: $\lim_{n\to\infty} \left| \frac{(x+1)^{n+1} \cdot n\,\cancel{(n+1)}}{\cancel{(n+1)}(n+2) \cdot (x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x+1)^n}(x+1)\,n}{(n+2)\cancel{(x+1)^n}} \right|$

$= |x+1| \cdot \lim_{n\to\infty} \frac{n \to 1}{n+2} = |x+1| < 1 \quad \therefore R = 1$

$-1 < x+1 < 1$

$-2 < x < 0$

Test Endpoints:

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ (conv.)

Alt Series Test

i) $\frac{1}{(n+1)(n+2)} < \frac{1}{n(n+1)}$ ∴ dec ✓

ii) $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$ ✓

(conv.)

Interval of Conv $[-2, 0]$

$x = 0$: $\sum_{n=1}^{\infty} \frac{(1)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$ (conv.)

Comp Test: $\frac{1}{n^2+n} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is p-series w/ $p = 2 > 1$ ∴ conv

23) Evaluate the integral $\int \frac{3}{1-x^4}\,dx$ by using the following two steps.

(a) Express the integrand as a power series. We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{3}{1-x^4} = 3 \cdot \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$

-OR-

$= 3 + 3x^4 + 3x^8 + 3x^{12} + \ldots$

(b) Integrate, term by term, the power series found in part (a).

$\int \frac{3}{1-x^4}\,dx = \int \sum_{n=0}^{\infty} 3x^{4n}\,dx$ -OR- $= \int \left[3 + 3x^4 + 3x^8 + 3x^{12} + \ldots \right] dx$

$= C + \sum_{n=0}^{\infty} \frac{3}{4n+1} x^{4n+1} = \left[3x + \frac{3}{5}x^5 + \frac{3}{9}x^9 + \frac{3}{13}x^{13} + \ldots \right] + C$

-OR-

24) Find the 3rd degree Taylor polynomial, T_3, for $f(x)=\frac{1}{x}$ centered at $a=1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\frac{1}{x}$	1
1	$-x^{-2} = \frac{-1}{x^2}$	-1
2	$2x^{-3} = \frac{2}{x^3}$	2
3	$-6x^{-4} = \frac{-6}{x^4}$	-6
4	$24x^{-5} = \frac{24}{x^5}$	24

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 - \frac{1}{1!}(x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3$$

$$T_3 = 1 - (x-1) + (x-1)^2 - (x-1)^3$$

— or —

$$T_3 = -x^3 + 4x^2 - 6x + 4$$

Please sign the honor statement below before submitting your exam.
I have neither given nor received unauthorized help on this exam.

Signed: ______________________________