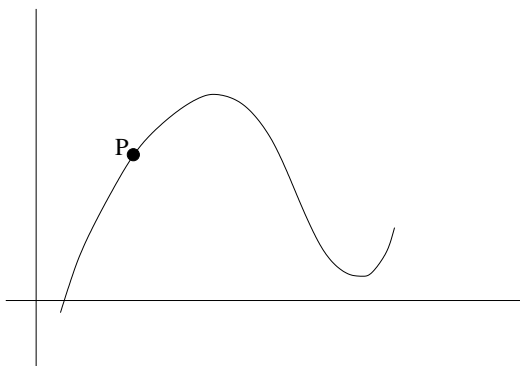


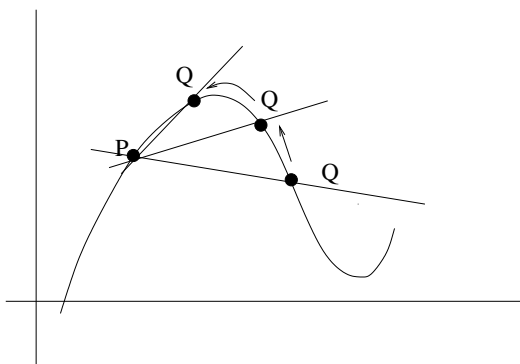
Exploring the Derivative

The derivative of a function is one of the most powerful tools in mathematics. It is often invaluable for investigations in both the physical, biological, and, to a lesser extent, the social sciences. In this lab, we introduce the concept of the derivative and explore this notion from the viewpoint of graphs, tables of values, and formulas. To help with this lab, you might want to use Module 2.1: “Secant Lines Approaching Tangent Lines” on the Tools for Enriching Calculus CD that came with your book.

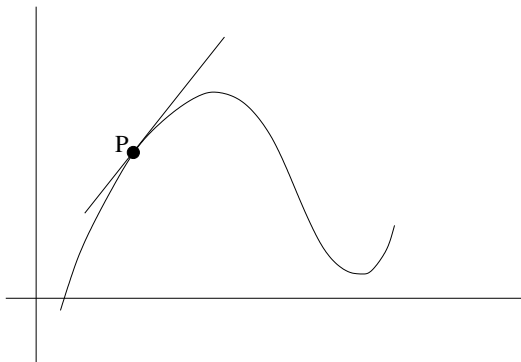
Introduction: We begin by defining the concept of a tangent line geometrically. Suppose that we have a function f and consider its graph $y = f(x)$:



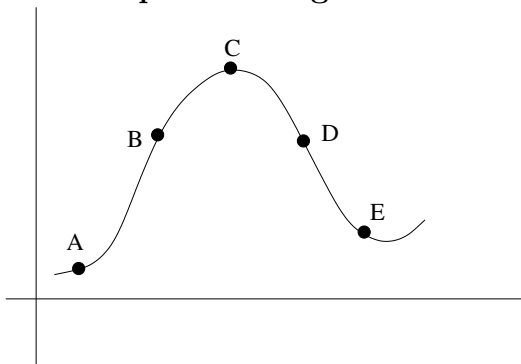
Now let P be a point on the given curve. We construct the tangent line of f as follows. Choose another point, say Q, that also lies on the curve, and connect the two points P and Q. This line PQ is called the secant line. We then let Q approach P along the curve, successively drawing the corresponding secant lines.



Then, in the limit as Q approaches P , we obtain the tangent line at the point P .



1. Examples of Tangent Lines. Here, we are given the graph of the function f .



Draw in the tangent lines to the graph of f at the indicated points.

Now we introduce an alternative standard terminology.

Definition of the Derivative (Geometric Version): Let f be a function and $P = (a, f(a))$ be a point on the graph of f . The *derivative* of f at the point P is defined to be the slope of the tangent line to the graph of f at the point P . According to long standing practice, the derivative of f at the point where $x = a$ (which corresponds to the point $P = (a, f(a))$) is denoted by $f'(a)$.

2. Derivatives and Graphs. The problems below refer to the graph given in the preceding exercise.

(a) At each point indicated, determine whether the derivative is positive, negative, or zero.

Derivative at point A is _____

Derivative at point B is _____

Derivative at point C is _____

Derivative at point D is _____

Derivative at point E is _____

(b) At which of the indicated points is the derivative the greatest and the least?

3. Derivatives and Tables of Values

In many physical situations, laboratory measurements produce data only at discrete points and we sometimes want to get estimates for derivatives from this data. This idea essentially means that we are going to have to use the slope of a secant, or perhaps of several secants, to estimate the slope of the tangent.

The Kelvin temperature of a fluid is an important indicator of its internal energy. Indeed, part of the definition of an ideal gas, over and above its equation of state, is that its internal energy is completely determined by its temperature. While real gases may not obey the ideal equation of state, their energy is often assumed to depend only on temperature. In a wide temperature range, carbon dioxide is an example of such a gas. Along a particular adiabatic path, we have the following data for internal energy of carbon dioxide as a function of temperature. The molar internal energy, u , is measured in Joules (J).

Temperature (°K)	u (J/mole)
290	6653
300	6939
310	7231

Table 1. Internal Energy as a Function of Temperature

a) Calculate the “left hand slope” of the secant that passes through the points at temperatures, 290°K and 300°K. Then calculate the slope of the secant through points at temperatures, 300°K and 310°K. The tangent line to this internal energy curve at $T = 300^\circ\text{K}$ has exact slope $du/dT(T = 300^\circ)$. Use the two secant slope values you worked out to estimate

the slope of this tangent.

Show your work in the space below.

b) According to the laws of thermodynamics, $du/dT = c_v$, where c_v is the molar heat capacity at constant volume. In Lab 1, the measured value of this specific heat was given as 28.90J/mole-°K. Compare your estimate for du/dT at 300°K with this measured value of c_v at 300°K .

c) The internal energy of a fluid is difficult to measure directly. The result is that *differences* between energies in Table 1 are physically valid, but any individual value of the molar internal energy, u , cannot be taken as a serious measurement of the energy of the gas. What would happen to your secant calculations if the molar internal energies above were shifted by adding, say, 350 Joules, to each?

4. Using the Tool Box.

As we have indicated in Lab 1, we often construct complicated functions by assembling parts from a supply of very simple functions. How do tangent lines behave when we do this? That is, we want to understand how the tangent line of a complicated function is related to the tangent lines of its simpler components. We will try three easy examples.

a) In the space below, sketch the graph of $f(x) = x^3$. Indicate on your graph the point corresponding to $x = 2$.

b) Let $c = 3$. Fill in the blank in the next sentence. The slope of the tangent to the scaled function, $y = cf(x)$ at the point corresponding to $x = 2$ is _____

(Your answer should involve the slope of the tangent to $y = f(x)$ at the point $x = 2$.)

The (correct) sentence above may be called the “scaling rule” for figuring tangent lines.

For the next rule, do the best you can to make a copy of the graph of $f(x) = x^3$ in the space below. Again, mark the point corresponding to $x = 2$.

For our other example, we consider an example in which a new curve is obtained by shifting the independent variable. That is, we now want to look at the graph of $y = f(x-1) = (x-1)^3$.

Fill in the blank in the next sentence: The slope of the tangent line to $y = f(x-1)$ at the point $x = 2$ is _____
(Your answer should involve a certain tangent line to $y = f(x)$, but NOT the tangent line at $x = 2$.)

Justify your answer geometrically. (It might help to make a sketch). .

If the sentence is completed correctly, one obtains the “sliding rule” for figuring tangent lines.

5. Derivatives and Formulas. The geometric description of the tangent line has an obvious appeal for its intuitive meaning. But for computational purposes, we need to translate what we have done into the language of algebra. The slope of a secant line is the ratio rise/run. In order to get a secant line through a point $(a, f(a))$ on a graph, we need a nearby point which we may obtain by adding a small number, h , to the argument a . Then the slope of the secant is the divided difference

$$\frac{f(a+h) - f(a)}{h}.$$

And now we get the slope of the tangent by driving h to 0 in this fraction.

Definition (Algebraic version): The derivative of a function f at a number a , denoted by $f'(a)$, is

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

provided that the limit exists.

We are now in a position to find tangent lines to some of our building block functions. Let us consider the function

$$f(x) = \frac{1}{x}.$$

Instead of working with numerical data from a hand calculator, we work symbolically with this function. For instance, to evaluate $f'(1)$, We should look at the divided difference

$$\frac{f(1+h) - f(1)}{h}.$$

which, in our case, is

$$\frac{\frac{1}{1+h} - \frac{1}{1}}{h} = -\frac{h}{(1+h)h} = -\frac{1}{1+h}.$$

What happens to this quantity when we drive h to 0? Quite obviously, the limiting value is -1.

Does this negative slope make sense graphically?

Repeat the steps of this calculation to work out the derivative of $y = f(x) = 1/x$ at an arbitrary value, say, $x = k$. Write out these steps in the space below.

It is part of a (somewhat confusing) notational style to write the result of this work as $f'(x) = \underline{\hspace{4cm}}$ (fill in the blank) in which the generic letter x replaces the letter k .

For another example, repeat this kind of calculation to work out the derivative of $g(x) = 1/x^2$ at an arbitrary value of x . $g'(x) = \underline{\hspace{4cm}}$.

And again, for $y = g(x) = 1/\sqrt{x}$ at an arbitrary value of the independent variable. Show the work below and write out a final formula,

$$g'(x) = \underline{\hspace{4cm}}.$$

6. Internal versus external pressure of a gas. If the volume of a gas is fixed and its temperature is permitted to vary, then the external pressure will, presumably, also need to vary according to the terms of the equation of state of the gas. By “external pressure”, we mean the pressure exerted by the gas on the walls of its container. This pressure is always denoted by the letter p .

The rate of change of external pressure with respect to temperature, $p'(T)$, is related to the idea of “internal pressure” of the gas. This latter quantity, which we will define below, is very difficult to measure directly. It is supposed to account for the claim that for many gases, there is an attractive force between pairs of individual molecules of the gas, a force which tends to reduce the external pressure. This force is explained by the mutual attraction felt by molecules whose balanced positive and negative charges are asymmetrically distributed in the geometry of that molecule. Water is an important example of such an asymmetry.

a) Take an ideal gas, with molar equation of state $p = RT/v$. Fixing the volume, compute the T -derivative of this function.

$$p'(T) = \underline{\hspace{4cm}}$$

b) Now an easy calculation. For the ideal gas, work out the product, $T \times p'(T)$.

$$T \times p'(T) = \underline{\hspace{4cm}}$$

c) For any gas, ideal or not, this function, $T \times p'(T)$ actually has a name. It is the *latent heat of volume expansion* and is denoted L_v . Express the latent heat of volume expansion of an ideal gas as a function of the pressure alone.

$$L_v \text{ (ideal)} = \underline{\hspace{4cm}}$$

d) Still using the ideal gas equation of state, compute the difference, $T \times p'(T) - p(T)$, a new function that chemists call the internal pressure of the gas. There seems to be no standard notation for this function. The internal pressure is typically quite small in comparison to the external pressure.

$$\text{Answer (Ideal gas): } T \times p'(T) - p(T) = \underline{\hspace{4cm}}$$

What would you say to someone who inquired about the force of mutual attraction between pairs of molecules of an ideal gas?

e) Calculate the latent heat of volume expansion for a van der Waals gas. (See the Chemistry Primer in the Appendix for the equation of state.) Don't forget, the volume is held constant so that the pressure of the gas is a function of the temperature alone.

$$L_v = T \times p'(T) = \underline{\hspace{4cm}}$$

f) Now calculate the internal pressure, $T \times p'(T) - p(T)$, for the van der Waals gas.

$$T \times p'(T) - p(T) = \underline{\hspace{10cm}}$$

g) How would you respond to someone who asked about the physical meaning of the term a/v^2 that appears in the equation of state for the van der Waals gas?

h) Use steps to calculate the internal pressure for a Bertholot gas at fixed volume. Do the same for a Redlich-Kwong gas. For reference, these equations of state are

$$\text{Bertholot: } p = \frac{RT}{v-b} - \frac{a}{Tv^2}$$

i) Bertholot: $T \times p'(T) = \underline{\hspace{10cm}}$

ii) Bertholot: $T \times p'(T) - p(T) = \underline{\hspace{10cm}}$

$$\text{Redlich-Kwong: } p = \frac{RT}{v-b} - \frac{a}{v(v+b)\sqrt{T}}$$

iii) Redlich-Kwong: $T \times p'(T) = \underline{\hspace{10cm}}$

iv) Redlich-Kwong: $T \times p'(T) - p(T) = \underline{\hspace{10cm}}$