

Inverse Functions

By now, you should have a good idea of what a function is. It is a rule (a description, table graph, or formula) that describes the unique relationship between two variables, one in terms of the other. When we see $y = f(x)$, we know that the function f takes an input x and gives us a single, unique, corresponding output y . In a sense, the function acts like a mathematical machine, converting one variable into another.

As an example, consider the familiar relationship between the Fahrenheit and Celsius temperature scales, $F = 1.8C + 32$, which we can write as $F = f(C)$. Here, the function f converts C (the input) into F (the output), and there is one, and only one, unique value of F for each value of C .

One typically uses this formula to ask for “the answer”, F , given “the problem”, C . But a frequent maneuver in mathematics, as in real-life, is to somehow elicit the problem from the answer. In other words, we want to reverse the original function to give C as a function of F . After some simple algebra, we would obtain:

$$C = \frac{F - 32}{1.8}$$

In the jargon of calculus, we call this the inverse function, written as $C = f^{-1}(F)$. Notice the connection: if $F = f(C)$, then $C = f^{-1}(F)$. Simply stated, the inverse function represents the math machine working in reverse; now F is the input, and C is the output. Pretty simple!

Warning: The notation f^{-1} refers to the inverse of f , not its reciprocal $1/f$.

In this lab project, we are going to look at the inverse function and some of its applications. You should be familiar with Section 1.6 of the textbook, and you might find the following concepts particularly useful:

- The definition of a one-to-one function and the horizontal line test on page 64.
- Definition of the inverse on page 64.
- Logarithmic functions and their properties.

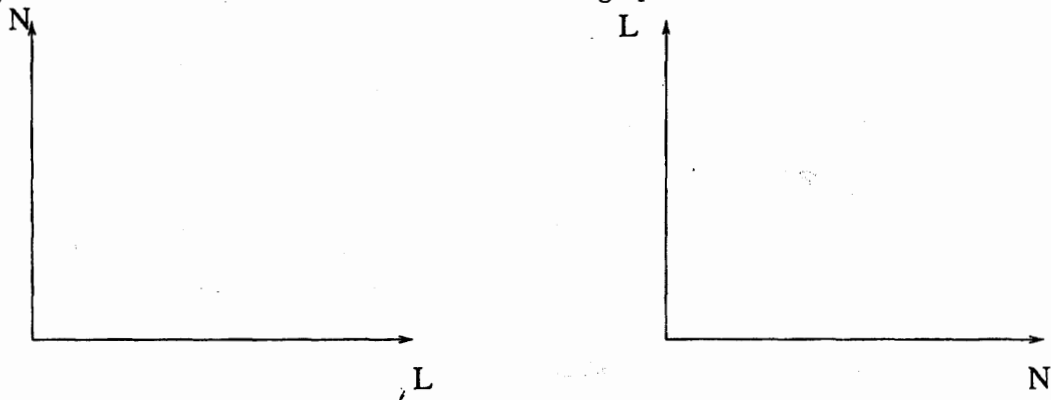
1. Simple Functions, Inverses, and Graphs

In the following two problems, you will be asked to develop simple functions, invert them, and look at their graphs. You should notice that the function and its inverse will be symmetric (by reflection) about the 45° line $y = x$. (Note that the textbook interchanges the variables after creating the inverse functions; this lab does not. The resulting graphs should look the same in both cases.)

Let N represent the time in New York, and L represent the time in Los Angeles, both in hours.

a) Develop a function $N = f(L)$, giving N in terms of L , and then invert it to find f^{-1} .

b) Plot the function and inverse on the graphs below (assume N and L vary from 0 to 24). Note that the axes are reversed in the second graph.



Now, consider a sphere of radius r and volume V .

c) Write V as a function of r , and then invert this function to give r as a function of V .

d) Plot the function and its inverse on side by side graphs, similar to parts a) and b) above.

3. A company reimburses its traveling sales representative \$160 per day for lodging and meals plus 35 cents per mile driven.

a) Construct a function giving daily cost, C , in terms of miles driven, m , and graph the function.

b) Question: How much will it cost the company if a sales representative drives 174 miles on a given day?

This company would like sales representatives to cover as much area as possible yet still keep their daily costs within reasonable limits. In other words, they are interested in how far a sales representative can travel for C dollars.

c) Use the concept of the inverse to develop a function that answers this and graph that function.

d) Inverse Question: If the company must limit this daily cost to \$200, how many miles, maximum, can a sales representative expect to travel?

Adiabatic¹ Expansion of a Gas

A gas is said to expand or contract adiabatically during some physical process if the gas is thermally insulated from its surroundings. Heating engineers often work in quite restricted temperature ranges in which they assume that a particular gas satisfies the ideal equation of state and that its two specific heats, c_p and c_v , are constant. The (constant) ratio of these two, c_p/c_v , is commonly denoted γ .

Then, during an adiabatic process,² the pressure, p , and molar volume, v , of the ideal gas must satisfy³

$$p = \frac{k}{v^\gamma}, \quad (1)$$

where k is some constant.⁴

4. Develop a formula that expresses the volume as a function of pressure during an adiabatic process.

5. Assume that equation (1) and the ideal gas law, $p = RT/v$, BOTH hold in the temperature range mentioned above. Use these two equations to work out the dependence of temperature on pressure (volume should not appear) along an adiabatic path. That is, find a function $T(p)$ that holds along an adiabatic path of an ideal gas with constant heat capacities.

6. For air (assuming it to be ideal), one takes $\gamma = 1.4$. To figure the constant k , we need to start somewhere. Take a single mole of air.⁵

a) What is its volume at a pressure of one atmosphere and temperature $T = 293.2^\circ\text{K}$? (Use the ideal gas law).

b) Use your answer to compute k for a mole of air that begins an adiabatic process at the given pressure and temperature. $k =$?

¹A-D-uh-Bat'-ik

²To be more technical about the terminology, one should say that the gas is undergoing a *reversible* adiabatic process.

³Johnston, et al, *Elements of Applied Thermodynamics*, 5th Ed., Naval Institute Press, 1992.

⁴The constant γ is an *exponent* in this formula.

⁵Air is not a pure substance. Hence, it does not have a molecular weight as such. By assuming that air is about 80% nitrogen (molecular weight 28) and 20% oxygen (molecular weight 32), it is common for engineers to say that the molecular weight of air is $.8 \times 28 + .2 \times 32$, about 29.

c) Using your value of k , sketch a graph of the volume of one mole of air as a function of pressure during an adiabatic process that begins at the given pressure and temperature. In the space below, make a sketch of the graph of this function, using, of course, the pressure as independent variable.

d) What volume is reached by one mole of air that starts at one atmosphere and $T = 293.2^\circ\text{K}$ and is expanded adiabatically to 0.5 atmospheres?

7. Inverting Difficult Functions.

In some earlier exercises, we found formulas for inverse functions. In many situations, we want to know that an inverse exists for a given function, even if it is difficult, if not impossible, to find a simple formula for that inverse. Suppose that, at a fixed temperature, say, $T = 260^\circ\text{K}$, we have a single mole of carbon dioxide.

a) Enter the van der Waals equation of state for CO_2 at this temperature as Y1 in your hand calculator. ⁶ Sketch the graph of Y1 below.

b) The question is, can one determine the volume as a function of pressure at this temperature? Why can't you uniquely determine this function? To answer this question,

⁶See the Chemistry Primer in the Appendix for the van der Waals equation of state.

show what such a $v(p)$ would look like graphically and discuss why this "function" is not a function.

c) For a single mole of carbon dioxide at $T = 260^\circ\text{K}$, use your graphing calculator to find possible volumes at $p = 10$ atmospheres, as predicted by van der Waals for the gas. Write these volumes below.

d) Do the same at 5 atmospheres.

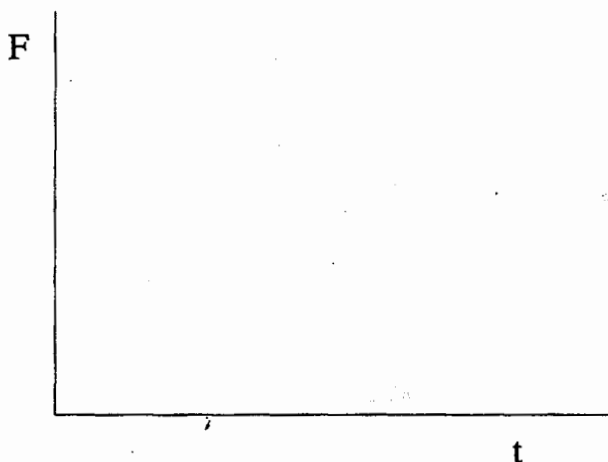
8. **Economics** With interest compounded monthly, the future value of a bank deposit grows according to the following function:

$$F = f(t) = D\left(1 + \frac{r}{12}\right)^{12t},$$

where D is the amount deposited, r is the nominal "decimalized" annual interest rate, F is the future value, D is the amount of the deposit (both in dollars), and t is the time in years.

a) If you deposit \$1200 in a bank at 3.6% interest ($r = 0.036$), develop a function that gives the time required for the deposit to reach a value F (i.e., write an expression for $F(t)$).

b) Graph the function.



c) How long will it take for that \$1200 deposit to grow to \$2400?

d) Write an expression for $f^{-1}(m)$, where m is money.

e) Sketch the graph on the same axes you used in part (b) and label it clearly as $f^{-1}(t)$.

9. **Forensic Science.** A murder is committed. At 8:04 pm, a forensic scientist investigating the crime is trying to pinpoint the time of death. She measures the temperature of the victim to be 92.4°F. One hour later, the body temperature is 89.4°F. According to Newton's Law of Heating and Cooling, body temperature would vary according to

$$T = T_{\infty} + (T_0 - T_{\infty})e^{-kt},$$

where t is the time in hours, T is the body temperature, T_{∞} is the constant ambient room temperature, T_0 is some known initial body temperature, and k is a constant. All temperatures are in °F.

a) If the room temperature at the crime scene was 70°F , solve for k in the Heating and Cooling equation given above. [Hint: you can do this by drawing values of T_0 , T and t from the data in the problem. You may assume that 8:04 pm corresponds to $t=0$ for this purpose.]

b) Find the time of death. Note that living humans have a body temperature of 98.6°F . [Hint: Now that you know k , you can start over with a different value for T_0 and an *unknown* value for t , the lapse of time between death and 8:04.