The College Of WILLIAM & MARY

Fall Term 2007

Math 111 Lab 1

New Functions from Old

In calculus, we tend to work with a fairly standard parts bin of functions, but we modify, transform, and combine these functions to fit a variety of problems and applications, much like a plumber fits standard pipes, elbows, and bends together to form a piping system. In this first lab project, we will look at the parts bin of functions used in calculus and develop a toolbox of methods and techniques to work with these functions. As we progress through Math 111, you will want these tools at your fingertips, so watch carefully!

1. Combining Functions

One of the most basic ways we can modify functions is to combine them through simple mathematical operations such as addition, subtraction, division, and multiplication. The nice thing about this tool is that it generally gives predictable results that make sense. For example, consider the two functions $f(x) = 2\cos(x)$ and $g(x) = x$.

a) Graph these functions on your calculator by entering $f(x)$ into Y1 and $g(x)$ into Y2. First, make sure your calculator is in radian mode by hitting the MODE button and selecting Radian if necessary. Then, you can use the Y= key to bring up the function register. Once the functions are typed in, you can use ZOOM-6:ZStandard to set your plot window to 10 in x and y. Sketch the result below.

b) Now, let's look at the combination $f(x) + g(x)$, sometimes written $(f + g)(x)$. The easiest way to do this is to enter Y3 = Y1 + Y2 into your calculator. You can use

VARS followed by 1:Function

to choose Y variables. Graph $f + g$ on your calculator, and sketch the result below. Does it make sense? Explain briefly.

c) Graph $f(x) - g(x)$, written $(f - g)(x)$, and sketch the result below. Interpret what you see.

d) Graph $f(x)g(x)$, sometimes written as $(fg)(x)$ (NOT $f(g(x))$).

e) Graph $f(x)/g(x)$. Sketch the result and mention any "problems" that pop up. In terms of the domain of the new function f/g, what does this mean?

2. **Translation: Shifts and Slides**

Visually, its easy to see how you could move functions around basically, all you need to do is redraw the graph with some sort of movement or translation. Algebraically, we can represent translation by breaking it into two separate motions one horizontal in x, and the other vertical in y. For $y = f(x)$, we can look at these translations in two ways:

i) The graph of $y = f(x) + c$
is the graph of $y = f(x)$ shifted upward if $c > 0$ and downward if $c < 0$.

ii) The graph of $y = f(x + c)$
is the graph of $y = f(x)$ slid to the left if $c > 0$ and to the right if $c < 0$.

In a more general sense, we can think of a function $y = a + f(x+b)$ where a and b control vertical and horizontal translation according to the rules above.

a) When an object is thrown at some angle, it follows a parabolic trajectory. Consider an object thrown from the origin at (0,0) that has the trajectory $y = f(x) = 4x - 0.9x^2$. Enter $f(x)$ into Y1 on your calculator, and graph it in a window of $0 \leq x \leq 10$ and $0 \leq y \leq 10$. Sketch the result below, labelling the trajectory A.

b) Now, let's assume the object was thrown from (0,4), or a height of 4, which means the trajectory will be shifted up by 4. Write an equation for the new trajectory below, plot the new trajectory on your calculator (enter it into Y2), and sketch the result on the graph above (label it curve B).

c) Instead of shifting the graph up, let's assume the object was thrown from the point (2,0) in other words, shifted to the right by 2. Write the equation of this trajectory below, graph it on your calculator under Y3, and sketch it on the graph above as curve C. Remember, shifting a graph to the right by an amount c means replacing x with $x - c$.

d) Find where the object lands for the trajectory in part (c) by setting $f(x) = 0$. You may find the program QUADRATIC helpful.

3. **Scaling and Reflecting Functions**

Besides translation, we can transform functions by scaling them and reflecting them about the x- or y-axis. As the name implies, scaling involves stretching or compressing a function, and we can do it vertically and horizontally as follows.

For the function $y = f(x)$ and the "scale factor" $c > 0$,

i) The graph of $y = cf(x)$
stretches the graph of $y = f(x)$ vertically if $c > 1$, and compresses it if $c < 1$.

ii) The graph of $y = f(cx)$
compresses the graph of $y = f(x)$ horizontally if $c > 1$ and stretches it if $c < 1$.

We can reflect or flip a function about the x- or y-axis as follows:

i) The graph of $y = -f(x)$
reflects the graph of $y = f(x)$ about (a mirror in) the x-axis.

ii) The graph of $y = f(-x)$
reflects the graph of $y = f(x)$ about the y-axis.

a) Consider the function $y = f(x) = x^3 - 2x^2$. Enter this into Y1 on your calculator, and graph it using the ZStandard window (in the ZOOM-menu). Sketch the result below, labelling the curve A.

b) Write the equation that stretches $f(x)$ horizontally by a factor of 3. Enter this into Y2, graph it, and sketch the result on the graph above as curve B.

c) Take the function you generated in part (b), stretch it vertically by a factor of 4, and reflect it about the y-axis. Write the new function below, enter it into Y3, and graph it. In the space below, sketch a graph of Y2 and Y3, labelling the curves B and C, respectively.

4. **Modeling the behavior of a gas.**
In this work, we need the reciprocal function, $p(v) = 1/v$.

a) Sketch the graph of this function on your hand calculator. Store this function as Y1.[The independent variable is v, the dependent is p.]

b) Chemists are only interested in this function for positive values of p and v. What are the most significant features of this portion of the graph?

Part of the description that chemists use for the behavior of a gas is its equation of state. Say we have 1 mole of the gas. The molar ideal equation of state is

$$p = \frac{RT}{v}, \tag{1}$$

where R is the universal gas constant, $R = 0.08206$ liter-atmospheres per °K, per mole.

Let us fix the temperature, say, at $T = 273.2$°K , which is about the temperature of freezing for water at 1 atmosphere of pressure, 0°C. Then $RT = 22.42$ liter-atmospheres per mole. (Notice how multiplication by T in degrees Kelvin removes that unit from the denominator of R).

c) Use $RT = 22.42$ as a scaling factor to show the graph of $p = (RT)(1/v)$. Store this graph as Y2. Display both Y1 and Y2 on your hand calculator. How do they compare with one another?

Next, we select a particular gas, say, carbon dioxide, CO_2. For a single mole of carbon dioxide, the dependence of wall pressure on temperature and volume does not quite fit the ideal. One explanation is that from a microscopic point of view, the appearance of the volume in the equation of state is supposed to capture the amount of space in which an individual carbon dioxide molecule may roam. The Dutch physicist, Johannes van der Waals, suggested that the molecules of carbon dioxide themselves had to occupy some minimal volume. He, therefore, proposed that a better model for the available roaming space would subtract the minimal volume from the geometric volume. The parameter $b = 0.04267$ liters represents this minimal volume for a single mole of carbon dioxide.

d) On the graphing calculator, show the graph of the third function,

$$p(v) = \frac{RT}{v-b},$$

at the fixed temperature $T = 260°K$.[1] Store this function as Y3. You will need to re-calculate RT to get the correct scaling factor.

e) Using a single coordinate frame, sketch the graphs of $p = RT/v$ and $p = RT(1/(v-b))$ at $T = 260°K$ in the space below. (Use window $.05 \leq x \leq 10$, $0 \leq y \leq 50$.)

Because the new equation of state still did not do a good job of representing the behavior of carbon dioxide, van der Waals went one step further in his microscopic analysis (and thereby won a Nobel prize) by making another correction. Thus, for one mole of carbon dioxide, he replaced the preceding version of p by subtracting a/v^2, where $a = 3.592$ liter^2-atmospheres. The full blown van der Waals equation of state for one mole of carbon dioxide is[2]

$$p = \frac{RT}{v-b} - \frac{a}{v^2}.$$

f) Using the numerical data for carbon dioxide, enter this function for $T = 260°K$ as Y4 in your hand calculator. Sketch the graph in the space below. (You will need to be careful about the range of volumes employed to see the important features of this function. Small volumes are crucial. Try a window $.05 \leq v \leq .6$ and $-14 \leq p \leq 50$.)

[1] For CO_2, this temperature is more interesting than 273.2°K.

[2] Later, we will see a physical interpretaion of the correction term, a/v^2.

g) How is this graph different from the graph for the ideal equation of state?

To qualify as ideal, a gas must first of all satisfy the ideal equation of state. In addition, it is required that its heat capacity at constant volume be independent of that volume. This peculiar sounding statement means that for one mole of an ideal gas, it takes the same amount of heat to raise its temperature, say, by 1°C, whether the single mole is in a quart jar or a 30 gallon beer barrel. But the heat capacity of an ideal gas is permitted to depend on temperature. Thus, it might require more (or less) heat to raise the temperature of an ideal gas from 300°K to 301°K than to raise its temperature from 1500°K to 1501°K. For a real gas, it is approximately true that its heat capacity at constant volume, c_v, is independent of volume. There are some standard models used to express the dependence of heat capacity on temperature. In the range between 300°K and 2000°K a quite good version for $c_v(T)$ is the cubic polynomial[3]

$$c_v(T) = aT^3 + bT^2 + cT + d. \tag{2}$$

Remembering that this heat capacity is measured in Joules/(mole-degree K) , what units would be appropriate for the constant d?

What units for the constant c?

What units for the constant a?

Some Data. It is much easier for chemists to do their laboratory work at constant pressure rather than at constant volume. When heating is done in this way, the corresponding function is called the heat capacity at constant pressure. It is commonly denoted c_p. In the table below, we give some measured data for both heat capacities for carbon dioxide.[4]

Temperature (°K)	c_v J/mole-°K)	c_p J/mole-°K)
300	28.90	37.22
500	36.31	44.62
1000	45.99	54.31
2000	52.03	60.35

i) Use the program HEAT on your hand calculator to work out the four coefficients[5] in

[3]The multipliers a and b in this formula have nothing to do with the constants a and b in the van der Waals equation

[4]Keenan,J. Chao,J., and Kaye, J. *Gas Tables*, 2[nd] Ed. John Wiley, 1983. p 103.

[5]HEAT uses *four* pieces of numerical data to determine the *four* polynomial coefficients.

formula (2) above for the heat capacity at constant volume of carbon dioxide. Write the cubic polynomial approximation for $c_v(T)$.

j) Using your formula, calculate $c_v(1600)$.[6]

5.Composite Functions

One additional way to combine functions is by composition, where one function becomes the input to another. For example, if we had $f(x) = \sin x$ and $g(x) = 1 + x^2$, we can combine these functions through the composition $f(g(x)) = \sin(g(x)) = \sin(1+x^2)$. The composition $f(g(x)$ is also written $f \circ g(x)$. A different composition would be $g(f(x) = g \circ f(x) = 1 + \sin^2 x$. Composite functions arise in any application where one function depends on another. As youll see in the next problem, composition can involve two or more functions.

One mole of an ideal gas [7] is contained in a circular cylinder that is closed at one end and equipped with a moveable piston at the other. The cylinder has radius 3 dm. While the gas is held at the fixed temperature $T = 293.2°$K the piston is slowly pulled out of the cylinder at the rate of 2 dm per second.[8]

If h is the distance from the inner surface of the piston to the base of the cylinder, then the volume of the cylinder is hA, where A is the area of the circular base. Initially, $h = 10$ dm.

a) Suppose the piston is moved outward at the rate of 2 dm per second. Write the volume of the cylinder as a function of t = time.

b) Remembering that the temperature of the ideal gas is held fixed while the piston is withdrawn, write the pressure, p, as a function of the volume v, and then as a function of the time, t.

c) How long does it take for the pressure to decrease to 50% of its original value?

d) How far has the piston moved when this pressure is reached? Label the units in your answer.

[6]Keenan, et al, give 50.57 J per mole per degree K for this value of c_v.

[7]See equation (1) above.

[8]A decimeter (dm) is 10 centimeters. Because a cubic centimeter is the same as a milliliter, a liter is the same as one cubic decimeter.