Expanding "Choice" in School Choice

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Abstract: Truthful revelation of preferences has emerged as a desideratum in the design of student assignment mechanisms in school choice programs. Gale-Shapley's deferred acceptance mechanism is strategy-proof for students but limits their ability to communicate their preference intensities. This results in ex-ante inefficiency when ties at school preferences are broken randomly. We propose a practical deferred acceptance mechanism which allows students, via signaling their preference intensities, to influence how they are treated in ties. It maintains truthful revelation of ordinal preferences and supports a greater scope of efficiency, a new notion that helps us to compare mechanisms on the efficiency ground.

Keywords: Gale-Shapley's deferred acceptance algorithm, choice-augmented deferred acceptance, tie breaking, ex ante Pareto efficiency.

1 Introduction

Public school choice has been subject to intense research and policy debate in recent years. Its goal of expanding one's choice of school beyond his/her residence area has broad public support, as exemplified by the number of districts that offer parental choice over public schools.[1] Yet, how to implement student assignment in a choice program remains actively debated.

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[1]Government policies that allow parents to choose schools for their children take various forms, including interdistrict and intradistrict public school choice, offered widely across the US, as well as open enrollment, tax credits and deductions, education savings accounts, publicly funded vouchers and scholarships, private voucher programs, contracting with private schools, home schooling, magnet schools, charter schools and dual enrollment. See an inter-

The debate, initiated by Abdulkadiroğlu and Sönmez (2003), is centered around a popular method, the "Boston" mechanism, which was used by Boston Public Schools (BPS) until the 2004-2005 school year to assign K-12 pupils to the city schools. Under the Boston mechanism, a student may increase her likelihood of being assigned to a school by ranking that school higher in her preference list during registration. This feature of the mechanism may tempt a student/parent to change her preference ordering of schools in the application form if it is likely to fail to get a place in her second choice school that would have been available had she listed that school as her first choice. In 2005, BPS replaced the Boston mechanism with a student-proposing deferred acceptance (DA) mechanism that simplifies parents' strategic choices by making it safe for them to state their true preference lists. Although many parents recognized the possibility of strategic manipulation of preferences under the Boston mechanism[2], some parents did see it as a merit, not as a shortcoming of the mechanism. Indeed, at a public hearing by the BPS School Committee, a parent argued:

> I'm troubled that you're considering a system that takes away the little power that parents have to prioritize... what you call this strategizing as if strategizing is a dirty word...[3]

As we will demonstrate, this argument is not without merits.

Student assignment in school choice is an application of the matching theory. A two-sided matching model (Gale and Shapley 1962) consists of a set of students and a set of schools. Every student has a preference relation over schools and every school has a preference relation over students. A pair of a student and a school *blocks* a matching if they are not matched with one another but each prefer to be matched with one another than to their partner in the matching. A matching of students to schools that respects capacity constraints at schools is *stable* if it is individually rational and there is no *blocking* pair of a student and a school (hereafter, we use "matching" and "assignment" interchangeably). Most of the existing theory on two-sided matching focuses on the case that all parties have strict preferences, mainly because indifferences in preferences were viewed as a "knife-edge" phenomenon in earlier applications like the entry-level labor markets for the American medical interns (Roth and Marilda Sotomayor, 1990). The DA mechanism is well-justified in terms of student incentives and welfare when student and school preferences do not involve indifferences. In particular, reporting preferences truthfully is a dominant strategy for every student under the DA mechanism (Dubins and Freedman, 1981; Roth, 1982), a property that we will refer to as *strategyproofness*. Furthermore, the DA algorithm produces a "unique" stable matching which every student weakly prefers to any other stable matching (Gale and Shapley, 1962), namely the *student-optimal stable matching*. By contrast,

active map at http://www.heritage.org/research/Education/SchoolChoice/SchoolChoice.cfm for a comprehensive list of choice plans throughout the US.

[2]See Abdulkadiroğlu, Pathak, Roth and Sönmez 2006 for evidence and further discussion.

[3]Recording from Public Hearing by the School Committee, 05-11-04

any stable matching may arise in a full-information Nash equilibrium of the Boston mechanism (Ergin and Sönmez, 2006).

However, a primary feature of the school choice problem is that there are indifferences/ties in how students are ranked by schools. For instance, in Boston, students who have siblings already attending a school are ranked by that school into a priority *class* that is higher than those of students who do not have any sibling at that school. In turn, a priority class contains many students with equal priority (hereafter we use "school preferences" and "school priorities" interchangeably). Despite the strong justification for the DA mechanism in case of strict preferences, indifferences in school preferences introduce a subtle trade-off between efficiency and incentives. In particular, without strict preferences, there may be multiple *student-optimal* stable matchings, each of which is not Pareto dominated by any other stable matching. More importantly, no strategy-proof mechanism selects a student-optimal stable matching for every preference profile (Erdil and Ergin, forthcoming). Furthermore, one has to break ties at school preferences in order to adopt the DA mechanism, and any inefficiency associated with a realized tie-breaking can not be removed ex post without harming students' incentives ex ante (Abdulkadiroğlu, Pathak and Roth, 2006).[4]

Yet, these negative conclusions, and the rest of the literature on school choice, are concerned with students' *ex post ordinal* welfare, namely, how well a given procedure assigns students based on their preference orderings for a *realized* tie-breaking. This perspective does not capture students' *ex ante welfare*, i.e., how well a procedure does on average across all realizations of tie-breaking, not just under some realization, and how efficiently it resolves students' conflicting interests based on their preference intensities, or their *cardinal utilities*.

To illustrate why ex ante welfare matters, suppose there are three students, $\{1, 2, 3\}$, to be assigned to three schools, $\{s_1, s_2, s_3\}$, each with one seat. All students are ranked into the same priority class at every school, and students' preferences are represented by the following von-Neumann Morgenstern (henceforth, vNM) utility values, where v^i_j is student i's vNM utility value for school s_j:

	v^1_j	v^2_j	v^3_j
$j=1$	4	4	3
$j=2$	1	1	2
$j=3$	0	0	0

Every feasible matching that assigns each student to a school is stable due to the indifferences at school preferences. Since the students have the same ordinal preferences, any such assignment is also ex post Pareto efficient, hence student-optimal stable. Therefore, there is no basis for comparing different procedures based on the ex post welfare criterion. In particular, the *stable improvement cycles* algorithm (Erdil and Ergin, forthcoming), which finds a student-optimal stable matching for every preference profile, has no bite in this example. Yet, how the students' conflicting interests are resolved matters greatly for their ex ante welfare.

[4] Also see Ehlers (2006) on matching with indifferences.

To elaborate this point, first note that, given a list of strict priorities, the DA algorithm works as follows: In the first step, every student applies to her most preferred school. Every school *tentatively* admits from the set of its applicants in the order of its priority order of students up to its capacity and rejects the remaining, who then apply to their next most preferred schools. In general, at every step, a school considers its new applicants and the students it has admitted tentatively. It tentatively admits from this set in the order of priority up to its capacity and rejects the remaining, who then apply to their next most preferred schools. Tentative assignments are finalized when there are no more rejections. To see the impact of tie-breaking in our example, suppose that the ties at school preferences are broken as follows: Assign each student a lottery number uniformly randomly. Whenever two students tie at a school, break the tie in favor of the student with the better lottery number. Then the DA assigns every students to each school with probability 1/3 since the students have the same ordinal preferences and the DA mechanism is strategy-proof, therefore all three submit their true ordinal preferences. Hence, the students obtain expected utilities of $EU_1 = EU_2 = EU_3 = \frac{5}{3}$.

It is easy to see that this assignment is ex ante Pareto-dominated by the following assignment: *Assign student 3 to* s_2*, and students 1 and 2 randomly between* s_1 *and* s_3, which yields expected utilities of $EU_1' = EU_2' = EU_3' = 2 > \frac{5}{3}$. Intuitively, starting from the random assignment, this latter assignment executes a trading of probability shares of schools by transferring student 3' share of schools 1 and 3 to students 1 and 2 in exchange of the latter students' shares of school 2. Such a trade is beneficial for all parties given their preference intensities.

The Boston mechanism works like the DA mechanism with the exception that assignments at every step are *final* under the Boston mechanism. Therefore, when students 1 and 2 submit their true preference rankings to the Boston mechanism, student 3 can secure s_2 by strategically ranking it as her first choice. In that case, students 1 and 2 apply to s_1 and student 3 applies to s_2 at the first step of the mechanism. Then student 3 gets *assigned* to s_2, whereas one of 1 and 2 is rejected by s_1. That student then applies to s_2. However, she is rejected by s_2 since 1's assignment at s_2 is *final*. This results in the more efficient matching above, which interestingly is the unique Nash equilibrium outcome of the Boston mechanism in our example.[5] The feature of the Boston mechanism crucial for this outcome is that a student can increase his probability of getting a school simply by ranking that school higher in his choice list. This ability to influence one's treatment in a competition is suppressed in the DA, for a school never discriminates its applicants based on where they rank that school in their choice lists. However, it is this latter property —nondiscrimination of applicants based on choice rankings—that yields the incentives for truthful revelation of preferences in DA. This suggests a tradeoff between incentives and ex ante efficiency. Clearly, the DA is extreme in resolving this tradeoff; it guarantees truthful revelation of preferences but denies students any "say" over how they should be treated by each school. Surprisingly, some parents seem to have found this feature of DA as troublesome, as one parent

[5]This does not contradict Ergin and Sönmez (2006)'s finding that the Boston mechanism is (weakly) Pareto dominated by the DA, which relies on strict preferences by the schools.

put it as follows at a BPS Public Hearing:

> ... if I understand the impact of Gale Shapley, and I've tried to study it and I've met with BPS staff... And in trying to understand this ... I thought I understood that in fact the random number [has] preference over your choices...[6]

The current paper suggests that there is a potentially better way to balance the tradeoff. Appreciable welfare gain can be obtained by offering students *simple* and *practical* ways to signal their preferences, with no sacrifice of strategy-proofness. We propose a practical procedure that accomplishes this goal and a new efficiency notion that enhances our understanding and ability to compare various assignment mechanisms on the efficiency ground. The next section illustrates our proposal.

2 Choice-Augmented DA Algorithm: Illustration

Suppose that schools' priorities are characterized by weak preferences. Then, ties must be broken to *generate* strict school preferences for the DA algorithm to be employed. There are two common methods of tie-breaking. *Single tie-breaking* randomly assigns every student a single lottery number to break ties at every school, whereas *multiple tie-breaking* randomly assigns a distinct lottery number to each student at every school. Clearly, a DA algorithm is well defined with respect to the strict priority list generated by either method. We refer the DA algorithms using single and multiple tie-breaking by *DA-STB* and *DA-MTB*, respectively.

We propose an alternative way to break a tie, one that allows students to influence its outcome based on their communication. The associated DA algorithm, which we refer as *Choice-Augmented Deferred Acceptance* (henceforth, CADA), is described as follows:

- **Step 1:** All students submit ordinal preferences, plus an "auxiliary message, "naming one's "target" school.

- **Step 2:** The schools' strict priorities over students are generated based on their *inherent priorities* and the students' auxiliary messages, as follows. First, each student is independently randomly assigned two lottery numbers. Call one *target lottery number* and the other *regular lottery number*. Each school's *strict priority list* is then generated as follows: (i) First consider the students in the school's highest priority group. Within that group, rank at the top those who name the school as their target. List them in the order of their target lottery numbers, and list below them the rest (who didn't name that school for target) according to their regular lottery numbers. (ii) Move to the next highest priority group, and list them below in the same fashion, and repeat this process until all students are ranked in a strict order.

[6] Recording from the BPS Public Hearing, 6-8-05

- **Step 3:** The students are then assigned to schools via the DA algorithm, using *each student's ordinal preferences* from Step 1 and *each school's strict priority list* compiled in Step 2.

To illustrate Step 2, suppose there are five students $N = \{1, 2, 3, 4, 5\}$ and two schools $S = \{A, B\}$, neither of which has inherent priority ordering over the students. Suppose students $1, 3$ and 4 named A for target and 2 and 5 named B for target, and that students are ordered according to their target and regular lottery numbers as follows:

$$\mathbf{T}(N) : 3 - 5 - 2 - 1 - 4; \quad \mathbf{R}(N) : 3 - 4 - 1 - 2 - 5.$$

Then the priority list for school A first reorders students $\{1, 3, 4\}$, who named that school as target, based on $\mathbf{T}(N)$, to $3 - 1 - 4$, and reorders the rest, $\{2, 5\}$, based on $\mathbf{R}(N)$ to $2 - 5$, which produces a complete list for A:

$$\mathbf{P}_A(N) = 3 - 1 - 4 - 2 - 5.$$

The priority list for B is determined as

$$\mathbf{P}_B(N) = 5 - 2 - 3 - 4 - 1.$$

The process of compiling the priority lists resembles the STB in that the same lottery is used by different schools, but only within each group. Unlike STB, though, different lotteries are used across different groups. This ensures that a student who has a bad draw at her target school gets a "new lease of life" with another independent draw for the other schools.[7]

Clearly, the deferred acceptance feature preserves stability and the incentives to reveal the ordinal preferences truthfully; the gaming aspect is limited to manipulating the outcome of tie-breaking. This limited introduction of "choice signaling" can however improve upon the DA rule in a significant way. In the above example, the CADA implements the Pareto superior matching: All students will submit the ordinal preferences truthfully, but 1 and 2 will choose s_1 as their target, and 2 will choose s_2 as her target. In this case, the CADA resembles the Boston mechanism.

In general, CADA is different from the Boston mechanism. In fact, if schools have many priorities (so their preferences are almost "strict"), then the auxiliary message would have little bite; thus the CADA will very much resemble the DA. Furthermore, CADA delivers a more efficient matching without sacrificing strategy-proofness. The rest of the paper makes this sense precise. That is, we demonstrate the nature of welfare benefits that CADA will have relative to the DA algorithms, when there are sufficiently numerous students and numerous school seats.

Specifically, we consider a model of a "large" economy populated by a continuum of students and a finite number of schools each with a continuum of capacities. We then compare alternative

[7] Although the primary reason for our choice is technical, this choice also has additional benefit of allaying a similar concern about the STB raised in the wake of the NYC redesign. One criticism against STB was that if a student has a bad draw, then she will not have a low priority with just one or two schools, but with *every* school she applies to. CADA mitigates this problem

procedures, DA-STB, DA-MTB and CADA, in terms of the scope of efficiency achieved under different procedures. To illustrate our approach, suppose a procedure determines for each student the probabilities of her getting assigned to alternative schools. Call these her *shares* of schools. Then the "scope of efficiency" can be measured by the set of schools whose shares cannot be traded among students in a way that benefits all the students. The bigger this set is, arguably the more efficient the outcome is, with the outcome being fully Pareto efficient if the set coincides with the entire set of schools.

Our main result is then stated in this term: *The CADA mechanism supports a greater scope of efficiency than the DA mechanisms with either tie-breaking procedure.* Specifically, the DA mechanisms supports efficient allocation of at most a pair of schools, whereas CADA supports efficient allocation of a (weakly) bigger set of schools. In particular, CADA entails efficient allocation among schools that are relatively popular—in the sense of being oversubscribed by students in their target choice. The economics of this property closely resembles that of competitive markets. Essentially, the students participating in CADA can be seen as making purchasing decisions on the shares of schools. For instance, a student can raise her share of a school, say A, by naming it for her target, but that lowers her priority standing in other schools, say B and C, thus reducing her shares of those schools. The exact tradeoffs faced by a student are determined by how many other students are picking A, rather than B or C, for their targets. If there are many such students, then raising a share of A is "expensive," for it requires giving up large amounts of shares of B and C. In other words, relative degrees of congestion at different schools act as "prices" that regulate individuals' decisions. In a "large" economy, students become price takers, so the resulting allocation resembles that of competitive markets, which, as is well known, yields an efficient allocation (among the oversubscribed schools).

In addition to showing the benefit of CADA, we also argue that DA-STB is more desirable than DA-MTB from an ex ante welfare perspective. In particular, we show that the former supports greater scope of efficiency than the latter. The choice between single versus multiple tie-breaking has proven to be an important policy choice in high school admissions in New York City.[8] Our finding informs the choice between DA-STB and DA-MTB in favor of the former.

The idea of CADA appears similar to the proposal by Sönmez and Ünver (2003) to imbed the DA algorithm in "course bidding"employed by some business schools. These two proposals differ in the application, however, as well as in the nature of the inquiry: We are interested in studying the benefit of adding a "signalling" element to the DA algorithm. By contrast, their interest is in studying the effect of adding ordinal preferences and the DA feature to the course bidding. In a broader sense, our paper is an exercise of mechanism design without monetary transfers, and in fact it is closer in nature to the recent ideas of "storable votes" (see Casella (2005)) and "linking decisions" (see Jackson and Sonnenschein, forthcoming).[9] Just like them, CADA "links" how a student is treated in a tie at one school to how she is treated in a tie at another school, and this linking makes communication credible. Clearly, applying the idea in a centralized

[8]See Abdulkadiroğlu, Pathak and Roth (2008) for a detailed discussion.

[9]See also Che and Gale 1998 and 2000 for the effect of budgetary limits in mechanism design

matching is novel and differentiates the current paper. There is a further difference. Jackson and Sonnenschein (forthcoming) demonstrated the efficiency of linking when (linkable) decisions tend to infinity, relying largely on the logic of the law of large numbers. To our knowledge, the current paper is the first to characterize the precise welfare benefit of linking a fixed (finite) number decisions (albeit with continuum of agents).

The rest of the paper is organized as follows. We present the formal model and welfare criterion in Section 3, provide welfare comparison across the three alternative procedures in Section 4. Section 5 presents simulation to quantify the welfare benefits of CADA. Section 6 then considers the implication of enriching the message used in the CADA and the robustness of our results to some students not behaving in a strategically sophisticated way. Section 7 concludes.

3 Model and Basic Analysis

3.1 Primitives

There are $n \geq 2$ schools, $S = \{1, ..., n\}$, each with a unit mass of seats to fill. There are mass n of students who are indexed by vNM values $\mathbf{v} = (v_1, ..., v_n) \in \mathcal{V} = [0,1]^n$ they attach to the n schools. The set of student types, $\mathcal{V}$, is equipped with a Lebesgue measure μ. We assume that μ is absolutely continuous with strictly positive density in the interior of $\mathcal{V} = [0,1]^n$ and that the values are distinct for μ-a.e. $\mathbf{v}$. That is, $\mu(\{\mathbf{v} \in \mathcal{V} | v_i = v_j \text{ for some } i \neq j\}) = 0$.

Let τ^k be any ordered list of any k schools, and let Π^k be the set of all such ordered lists of k schools, with their union denoted $\Pi := \cup_{k=1}^n \Pi^k$. Let $\pi^k(\mathbf{v})$ be the type $\mathbf{v}$-student's k most preferred schools (listed in the descending order of her preferences). Let $m_{\tau^k} := \mu(\{\mathbf{v} \in \mathcal{V} | \pi^k(\mathbf{v}) = \tau^k\}$ be the measure of students whose ordinal preferences are τ^k. By the full support assumption, $m_{\tau^k} > 0$ for each $\tau^k \in \Pi^k \ \forall\ k \leq n$. Hence, for each $i \in S$, $m_i = \mu(\{\mathbf{v} \in \mathcal{V} | \pi^1(\mathbf{v}) = i\})$ represents the measure of students whose most preferred school is i. It is useful to define the set

$$S^{**} = \{i \in S | m_i \geq m_j \forall j \in S\}$$

of *most popular* school(s).

Finally, let $\mathfrak{m} := \{m_\tau\}_{\tau \in \Pi^n}$ be a profile of measures of all ordinal types. Let $\mathfrak{M} := \{\{m_\tau\}_{\tau \in \Pi^n} | \sum_{\tau \in \Pi^n} m_\tau = n\}$ be the set of all possible measure profiles. The set $\mathfrak{M}$ has $n! - 1$ dimensions. We say a property holds *generically* if it holds for a subset of $\mathfrak{m}$'s that has the same Lebesque measure as $\mathfrak{M}$.

The school choice procedures we consider involve random tie breaking. It is convenient to explicitly model the randomizing device used to break the ties. For our purpose, it is sufficient to consider a vector $\omega = (\omega_1, ..., \omega_n) \in [0, n]^n =: \Omega$ of uniformly and independently generated numbers. (The vector of ω will be sufficiently rich enough to model the procedures we study.) Formally, we augment the type space by incorporating the random draw to $\mathcal{V} \times \Omega =: \Theta$, with its generic element denoted $\theta := (\mathbf{v}, \omega)$, and endow it with a product measure $\eta = \mu \times \xi_1 \times ... \times \xi_n$, where ξ_i is a uniform measure satisfying $\xi([0, \omega_i]) = \frac{\omega_i}{n}$ for each $\omega_i \in [0, n]$. This formalism avoids appealing to the law of large numbers (on the continuum of agents), by ensuring that a fraction

$\frac{\omega_i}{n}$ of the student mass draws ω_i or less on each i-th random variable. A student of type $\theta = (\mathbf{v}, \omega)$ is then interpreted as having values $\mathbf{v}$ and drawing a vector ω. The student never observes ω, so her action required by the procedure will be measurable with respect to only $\mathbf{v}$; whereas (part or all of) ω component is "discovered" by the schools for their use in tie-breaking.

An *assignment*, denoted by $\mathbf{x}$, is a probability distribution over S, and this is an element of a simplex, $\Delta := \{(x_1, ..., x_n) \in \mathbb{R}^n_+ | \sum_{i \in S} x_i = 1\}$. An *ex post allocation* is a measurable function $\psi := (\psi_1, ..., \psi_n) : \Theta \mapsto \Delta$ such that $\psi_i(\theta) \in \{0, 1\}$ and that $\int \psi_i(\theta)\eta(d\theta) = 1$ for each $i \in S$. Namely, ψ assigns a student with $\mathbf{v}$ to school j upon drawing ω such that $\psi_j(\mathbf{v}, \omega) = 1$. Let $\mathcal{Y}$ be the set of all ex post allocations. Later, we shall describe how each procedure generates an ex post allocation. Some procedures may not use the entire vector of ω, so the ex post allocation they produce may be measurable with respect to only some components of ω.

We are primarily interested in how a procedure determines the assignment for each student ex ante prior to conducting the lottery. To this end, we define *ex ante allocation* to be a measurable function $\phi := (\phi_1, ..., \phi_n) : \mathcal{V} \mapsto \Delta$ such that $\int \phi_i(\mathbf{v})d\mu(\mathbf{v}) = 1$ for each $i \in S$, with the interpretation that student $\mathbf{v}$ is assigned by mapping $\phi = (\phi_1, ..., \phi_n)$ to school i with probability $\phi_i(\mathbf{v})$. Let $\mathcal{X}$ denote the set of entire ex ante allocations. An ex post allocation ψ *induces* an ex ante allocation via $\phi(\mathbf{v}) := \int_\omega \psi(\mathbf{v}, \omega)\xi(d\omega)$.

3.2 Welfare Standards

An ex post allocation $\tilde{\psi} \in \mathcal{Y}$ *weakly Pareto-dominates* ex post allocation $\psi \in \mathcal{Y}$ if

$$\mathbf{v} \cdot \tilde{\psi}(\mathbf{v}, \omega) \geq \mathbf{v} \cdot \psi(\mathbf{v}, \omega) \text{ for } \eta - a.e. (\mathbf{v}, \omega), \tag{1}$$

and that $\tilde{\psi} \in \mathcal{Y}$ *Pareto-dominates* allocation $\psi \in \mathcal{Y}$ if (**??**) holds and there exists a set $A \subset \Theta$ with $\eta(A) > 0$ such that

$$\mathbf{v} \cdot \tilde{\psi}(\mathbf{v}, \omega) > \mathbf{v} \cdot \psi(\mathbf{v}, \omega) \text{ for all } (\mathbf{v}, \omega) \in A.$$

Pareto domination is defined analogously for ex ante allocations with respect to measure μ. Pareto optimality is then defined.

Definition 1. *(i) An allocation $\psi \in \mathcal{Y}$ is **ex post Pareto optimal (EPPO)** if there is no other allocation in $\mathcal{Y}$ that Pareto-dominates ψ.*[10]

*(ii) An allocation $\phi \in \mathcal{X}$ is **ex ante Pareto optimal (EAPO)** if there is no other allocation in $\mathcal{X}$ that Pareto-dominates ϕ.*

[10] As mentioned, different procedures may use different tie-breaking procedures, using some or all components of ω. There is a question as to whether the set of allocations $\mathcal{Y}$ must be restricted to be measurable to the same components of ω as the candidate allocation being tested for Pareto optimality. This issue does not matter, however, since the measurability with respect to even a single component of ω gives as much flexibility as measurability with respect to the entire vector of ω, in the sense that if an allocation measurable with respect to the entire vector of ω Pareto dominates a given allocation, there is one measurable with respect to only one component of ω that also Pareto dominates the candidate allocation.

It is useful for our purpose to introduce a weaker property. As mentioned earlier, we will characterize the scope of markets that support Pareto efficiency. To begin, fix an assignment $\mathbf{x} \in \Delta$, and a subset $K \subset S$ of schools. An assignment $\tilde{\mathbf{x}} \in \Delta$ is said to be a *within K reassignment of* $\mathbf{x}$ if $\tilde{x}_j = x_j$ for each $j \in S \backslash K$, and let $\Delta^K_{\mathbf{x}} \subset \Delta$ be the set of all such reassignments. Then, a *within K ex post reallocation of* an allocation $\psi \in \mathcal{Y}$ is an element of a set

$$\mathcal{Y}^K_\psi := \{\tilde{\psi} \in \mathcal{Y} | \tilde{\psi}(\theta) \in \Delta^K_{\psi(\theta)}, \text{ a.e. } \theta \in \Theta\},$$

and a *within K ex ante reallocation of* an allocation $\phi \in \mathcal{X}$ is an element of a set

$$\mathcal{X}^K_\phi := \{\tilde{\phi} \in \mathcal{X} | \tilde{\phi}(\mathbf{v}) \in \Delta^K_{\phi(\mathbf{v})}, \text{ a.e. } \mathbf{v} \in \mathcal{V}\}.$$

In words, a within-K reallocation of ϕ involves possible trading of students' assignment probability shares over schools within K.

Definition 2. *(i) For any $K \subset S$, an allocation $\psi \in \mathcal{Y}$ is **ex post Pareto optimal (EPPO) within** K if there is no within K ex post reallocation of ψ that Pareto dominates ψ.*

*(ii) For any $K \subset S$, an ex ante allocation $\phi \in \mathcal{X}$ is **ex ante Pareto optimal (EAPO) within** K if there is no within K ex ante reallocation of ϕ that Pareto dominates ϕ.*

*(iii) An ex ante (resp. ex ante) allocation is **pairwise EAPO** (resp. **EPPO**) if it is EAPO (resp. EPPO) within every $K \subset S$ with $|K| = 2$.*

These welfare criteria are quite intuitive. Suppose the students are initially endowed with ex ante shares ϕ of schools but they can trade these shares amongst them. Can they trade mutually beneficially if the trading is restricted involve only shares of subset K? The answer is no if allocation ϕ is *Pareto optimal within K*. In other words, the size of the latter set represents the restriction on the trading technologies and thus determines the scope of markets within which efficiency arises. The bigger the set is, the less restricted the agents are in the scope of trading, so the Pareto optimality within a bigger set means a more efficient allocation. Clearly, if an allocation is Pareto optimal within the set of all schools, then it is fully Pareto optimal. In this sense, we can view the size of such set as a measure of efficiency. Some obvious observation follows.

Lemma 1. *(i) If an allocation is EPPO (resp. EAPO) within K', then it is is EPPO (resp. EAPO) within $K \subset K'$;*

(ii) An ex post allocation $\psi \in \mathcal{Y}$ is EPPO within $K \subset S$ if the ex ante allocation it induces, $\phi(\cdot) = \int \psi(\cdot, \omega)\xi(d\omega)$, is EAPO within K.

(iii) For any K with $|K| = 2$, if an ex post allocation is EPPO within K, then the ex ante allocation it induces is EAPO within K.

(iv) If an ex post allocation is EPPO, then the ex ante allocation it induces is pairwise EAPO.

These characterizations are tight. The converse of Part (ii) does not hold for any K with $|K| > 2$. For any $n \geq 3$, an EPPO allocation can induce an ex ante allocation that is not even EAPO within the set of three schools. This can be seen by extending the example in the introduction, by adding any number of schools that all students rank uniformly and below the top three. Uniform ordinal preferences mean that any ex post allocation would be EPPO. But random allocation is not EAPO within the set of the top three schools.[11] Likewise, an ex ante allocation can be EAPO within K, but the ex post allocation that induces it may not be EPPO within any $K' \supsetneq K$. To see this, imagine a situation in which an ex post allocation is Pareto improvable upon only via a trading cycle (involving positive measure of students) that includes a school in $K' \backslash K$. In that case, the allocation can be EAPO within K yet it will not be EPPO within K'.

3.3 Alternative School Choice Procedures

We consider three alternative procedures for assigning students to the schools: (1) Deferred Acceptance with Single Tie-breaking (DA-STB), (2) Deferred Acceptance with Multiple Tie-Breaking (DA-MTB), and (3) Choice-Augmented Deferred Acceptance (CADA). These procedures, introduced earlier, can be extended to the continuum of students in a natural way.

The alternative procedures differ only by the way the schools break ties. Formally, we introduce a tie-breaker function which determines the priority for each student as a function of the random draw (as well as their auximiliary message in the case of CADA), in the event of a tie. Formally, a *tie-breaker function for school* i is a bounded measurable function $F_i : \Theta \mapsto \mathbb{R}$, such that a student θ' is interpreted as having a higher priority than student θ if $F_i(\theta') < F_i(\theta)$. A *tie-breaker* is a profile $\mathcal{F} = \{F_i : i \in S\}$ of tie-breaker functions. Specifically, the tie-breakers for DA-STB, DA-MTB, and CADA are determined as follows:

- *DA-STB:* The STB rule uses the same tie-breaker function for all schools. This is modeled by a tie-breaker with

$$F_i(\mathbf{v}, \omega_1, ..., \omega_n) = \omega_1,$$

  $\theta = (\mathbf{v}, \omega_1, ..., \omega_n)$, for *every* school $i \in S$. In other words, a draw's draw ω_1 serves as a priority number for all schools. Heuristically, a real number ω_1 is drawn randomly from an interval $[0, n]$, for each student, which then serves as her priority score.[12]

- *DA-MTB:* The MTB rule produce a randomly and independently drawn priority list for each school. This is modeled by a tie-breaker, with

$$F_i(\mathbf{v}, \omega_1, ..., \omega_n) = \omega_i,$$

[11] Even though the example featured the finite case. The outcome can be readily extended to the continuum of students model.

[12] This heuristics invokes a law of large numbers, but our formal method does not rely on it for we assume a well-behaved randomization device.

for $i \in S$ and for each $\theta = (\mathbf{v}, \omega_1, ..., \omega_n)$. In other words, for each student, a vector $(\omega_1, ..., \omega_n)$ of independent draws determines her priority scores at different schools.

- *CADA:* In CADA, each student sends an auxiliary message of a target school (in addition to their ordinal preferences over schools). Given a (measurable) strategy profile $s : \mathcal{V} \to S$ determining the auxiliary message for each intrinsic type $\mathbf{v}$, the tie-breaker function for school i is given by
$$F_i(\mathbf{v}, \omega_1, ..., \omega_n) = \begin{cases} \omega_1 & \text{if } s(\mathbf{v}) = i \\ n + \omega_2 & \text{if } s(\mathbf{v}) \neq i \end{cases}$$
That is, under F_i, ties are broken first in favor of students who report i as their target school, within them according to the random draw ω_1, and then ties among the rest are broken according to a random draw $\omega_2 + n$ (where n act as a "penalty score" n). Clearly, F_i is a measurable function since ω_1 and s are measurable.

For each procedure, the DA algorithm can be readily defined using the appropriate tie-breaker and the students' ordinal preferences as inputs. Appendix A provides a precise algorithm, which is sketched here. At the first step, each student applies to her most preferred school. Every school i tentatively admits up to unit mass from its applicants in the order of its priority order, and reject the rest if there is any. In general, each student who was rejected in the previous step applies to her next preferred school. Each school considers the set of students it has tentatively admitted and the new applicants. It tentatively admits up to unit mass from these students in the order of its priority, and rejects the rest. The process converges when the set of students that are rejected has zero measure. Although this process might not complete in finite time, it converges in limit, and the allocation in the limit is well defined (see Theorem **??** of Appendix A). Further, each of the procedure is ordinally strategy proof:

Theorem 1. (Ordinal strategy-proofness) *In each of the three procedures, it is a (weak) dominant strategy for each student to submit her ordinal preferences truthfully.*

Proof: The proof follows from Theorem **??** in Appendix A.

3.4 Characterization of Equilibria

□ *DA-STB* and *DA-MTB*

The DA-STB process induces a cutoff $c_i \in [0, n]$ for each school i such that a student who ever applies to school i gets assigned to that school if and only if her (single) draw ω is less than c_i. The existence of such a cutoff follows from the fact that an applicant with a lower draw can never be rejected by a school in favor of another applicant with a higher draw. Therefore, in equilibrium each school has a cutoff in $[0, n]$, and the n cutoffs can be listed in an ascending order as $(\hat{c}^1, ..., \hat{c}^n)$, with $\hat{c}^i \leq \hat{c}^j$ for $i < j$. (The order may be weak since the cutoff may be the same for several schools, in which case we simply repeat the same number.) It is useful to establish that these cutoffs are uniquely determined by $\mathfrak{m} := \{m_\tau\}_{\tau \in \Pi^n}$ and all distinct generically:

Lemma 2. *The cutoffs for the schools under DA-STB are uniquely determined by* $\mathfrak{m}$*, and satisfies* $\hat{c}^1 > 0$ *and* $\hat{c}^n = n$*. The most popular school(s) in* S^{**} *have the lowest cutoff* $\hat{c}^1$*. For a generic* $\mathfrak{m}$*, the cutoffs are all distinct.*

DA-MTB is similar to DA-STB, except that each student has independent draws $(\omega_1, ..., \omega_n)$, one for each school. The DA process again induces a cutoff $c_i \in [0, n]$ for each school i such that a student who ever applies to school i gets assigned to it if and only if her draw for school i, ω_i, is less than c_i. While the cutoffs under DA-STB are typically different from those under DA-MTB, we must have at least one school whose cutoff is equal to n.[13] We call such a school the "worst" school. (There is no presumption that the worst school under DA-STB is also the worst school under DA-MTB.)

□ *CADA*

As with the two other procedures, given the students' strategies on their messages, the DA process induces cutoffs for the schools, one for each school in $[0, 2n]$. Of particular interest is the equilibrium in the students' choices of messages. Given Theorem **??**, the only nontrivial part of the students' strategy concerns her "auxiliary message." Let $\nu = (\nu_1, ..., \nu_n) : \mathcal{V} \mapsto \Delta$ denote the students' mixed strategy, whereby a student with $\mathbf{v}$ "names school i for her target" with probability $\nu_i(\mathbf{v})$. We first establishes existence of equilibrium.

Theorem 2. (Existence) *There exists an equilibrium in pure strategies.*

We say that a student *applies to school* i if she is rejected by all schools she lists ahead of i in her (truthful) ordinal list. We say that a student *subscribes to school* $i \in S$ if she picks school i for her target *and* applies to that school during the DA process. (The latter event depends on where she lists school i in her ordinal list and the other students' strategies as well as the outcome of tie-breaking). Let $\bar{\nu}_i^*(\mathbf{v})$ be the probability that a student $\mathbf{v}$ subscribes to school i in equilibrium. We say a school $i \in S$ is *oversubscribed* if $\int \bar{\nu}_i^*(\mathbf{v}) d\mu(\mathbf{v}) \geq 1$ and *undersubscribed* if $\int \bar{\nu}_i^*(\mathbf{v}) d\mu(\mathbf{v}) < 1$. In equilibrium, there will be at least (generically, exactly) one undersubscribed school which anybody can get in, if she fails to get an any other schools she lists ahead of that school. Formally, a school $w \in S$ is said to be "worst" if its cutoff on $[0, 2n]$ equals precisely $2n$. Then, we have the following lemma.

Lemma 3. *(i) Any student who prefers the worst school the most is assigned to that school with probability 1 in equilibrium. (ii) If her most preferred school is undersubscribed but not the worst school, then she names that school for her target in equilibrium. (iii) For almost every student with* $\mathbf{v}$ *such that* $\pi_1(\mathbf{v}) \neq w$*,* $\nu^*(\mathbf{v}) = \bar{\nu}^*(\mathbf{v})$ *in equilibrium.*

In light of Lemma **??**-(iii), we shall refer to "picks a school i as target" simply as "subscribes to school i."

[13] If all cutoffs are strictly less than n, then there is a positive measure of students who are never assigned to any school. There must be a school with its capacity unfilled, so that school's cutoff cannot be less than n, a contradiction.

4 Welfare Analysis of Alternative Procedures

We compare ex ante welfare of three alternative procedures in this section. We begin with DA-STB.

Theorem 3. (DA-STB) *(i) In the DA-STB allocation, the most popular school $i \in S^{**}$ is assigned only to the students who prefer that school the most.*

(ii) The allocation from DA-STB is EPPO and is thus pairwise EAPO.

(iii) For a generic $\mathfrak{m}$, there exists no $K \subset S$ with $|K| > 2$ such that ϕ^S is EAPO within K.

The DA-STB coincides with the random serial dictatorship, and it is well known that the latter produces an ex post efficient allocation. The intuition can be seen in our continuum of students model in the following way. Suppose the schools are listed in the ascending order of their cutoff numbers, $S = \{s_1, ..., s_n\}$, where school s_i has a lower cutoff (and is thus more popular) than school s_j if $i < j$. The cutoffs are depicted as follows:

Figure 1: Ex post Pareto efficiency of DA-STB

The STB has an interesting feature. Any students assigned to school s_i must have a draw $\omega_1 \leq c_{s_i}$, which means that the student must have listed that school ahead of every school s_j with $j > i$ (or else that student could never have been assigned to s_i). It follows from ordinal strategyproofness that such a student prefers s_i to all other schools with higher cutoff numbers. This means the following: Any students who are assigned to s_1 (the most popular school) must prefer it the most (as is claimed in Part (i)). Hence, these students can never be better off from being reassigned. Working recursively then, assuming that no students assigned to schools, $s_1, ..., s_{i-1}$, are reassigned, no students assigned to school s_i can never be better off from being reassigned. This shows the ex post Pareto efficiency. The pairwise ex ante Pareto optimality is then a direct consequence of the ex post efficiency (by Lemma **??**-(iii)).

Most important of all, according to Part (iii), this is the most you can get from DA-STB. In otherwise, DA-STB produces no better ex ante efficiency performance than is explainable by its ex post efficiency. To see Part (iii), recall from Lemma **??** that the schools' cutoffs are generically distinct. Take any set $\{i, j, k\}$ with $c_i < c_j < c_k$.

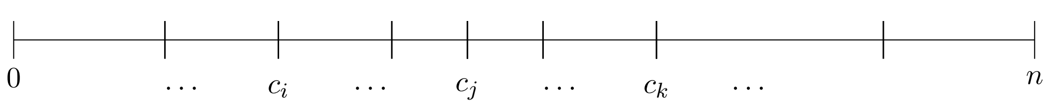

Figure 2: Ex ante Pareto inefficiency of DA-STB within $\{i, j, k\}$.

Then, by the full support assumption, there exists a positive measure of $\mathbf{v}$'s satisfying $v_i > v_j > v_k > v_l$ for all $l \neq i, j, k$. These students will then have the positive chance of being assigned to each school in $\{i, j, k\}$, for their draws will land in the intervals, $[0, c_i]$, $[c_i, c_j]$ and $[c_j, c_k]$, with positive probabilities. Again, given the full support assumption, such students will all differ in their marginal rate of substitution among the three schools. Then, just as with the motivating example, one can construct a mutually beneficial trading of shares of these schools among these students.

Next, consider the DA-MTB. Let school $w \in S$ be the worst school if its cutoff under DA-MTB is n. There exists only one worst school for a generic $\mathfrak{m}$.

Theorem 4. (DA-MTB) *(i) For $n \geq 3$, a positive measure of seats at each school are assigned to some students who do not prefer that school the most.*

(ii) The allocation from DA-MTB is EAPO within $\{i, w\}$ for each $i \in S \backslash \{w\}$.

(iii) Generically, there exists no $K \subset S \backslash \{w\}$ with $|K| > 1$ such that the (resp. ex ante) allocation from DA-MTB is EPPO (resp. EAPO) within K.

(iv) For a generic $\mathfrak{m}$, there exists no $K \subset S$ with $|K| > 2$ such that the ex post (resp. ex ante) allocation from DA-MTB is EPPO (resp. EAPO) within K.

The main results follow from the failure of ex post Pareto efficiency in the DA-MTB. This can be explained as follows. Take any two schools $\{i, j\}$ each of which is not a worst school. Type i student may have a good draw at school j but a bad draw at school i (e.g., (ω_i, ω_j) in Figure 3); and the opposite may occur to a type j student (e.g., (ω_i', ω_j')).

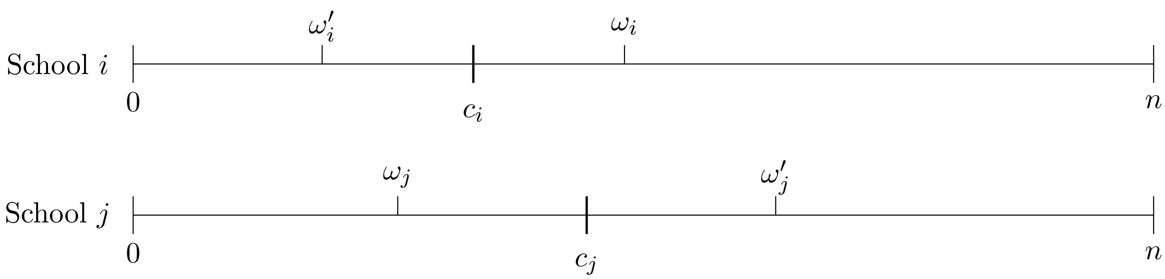

Figure 3: Pareto inefficiency within $\{i, j\}$ under DA-MTB.

In this case, the former will be assigned to j and the latter to i, so both students will benefit from swapping their assignments. This results in the failure of pairwise ex ante Pareto optimality. To see Part (ii), suppose school $j = w$ is the worst school. Then, $c_j = n$. This means that any students assigned to school i must prefer that school over school w. Hence, they can never be better off from any within-$\{i, w\}$ ex post reassignment, so the allocation is EPPO within $\{i, w\}$, and in turn EAPO within $\{i, w\}$ (by Lemma **??**-(iii)).

Theorems **??** and **??** have an obvious implication.

Corollary 1. *Assume $n \geq 3$. Then, for a generic $\mathfrak{m}$, an EAPO allocation never arises from the DA algorithm with either tie-breaking procedure.*

We next turn to the CADA algorithm. The welfare properties of its allocation are characterized as follows.

Theorem 5. (CADA) *(i) Any equilibrium allocation of CADA is EPPO, and is thus pairwise EAPO.*

(ii) Any equilibrium allocation of CADA is EAPO within the set of oversubscribed schools.

(iii) If all but one schools are oversubscribed, then the equilibrium allocation of CADA is EAPO.

Theorem **??**-(ii) and (iii) showcase the ex ante efficiency benefit associated with CADA. As mentioned earlier, the benefit parallels that of a competitive market. Essentially, CADA supports "competitive markets" for the shares of oversubscribed schools. Each student is given a "budget" of unit probability she can allocate across alternative schools with the naming of a target school. A given unit probability can buy different amounts of shares for different schools, depending on how many others name those schools. If a mass $z_i \geq 1$ students applies to school i, allocating a unit budget can only buy a share $1/z_i$. Hence, the relative congestion at alternative schools, or their relative popularity, serves as relative "prices" for these schools. In a large economy, individual students take these prices as given, so the prices play the usual role of allocating resources efficiently. It is therefore not surprising that the proof follows the First Welfare Theorem.

Why are competitive markets limited only to oversubscribed schools? Why not undersubscribed schools? The reason has to do with that one can get into an undersubscribed school in two different ways: She can name it for her target, in which case she gets assigned to it for sure; alternatively, she can name an oversubscribed school but the school rejects her, in which case she may still get assigned to the undersubscribed school via the usual DA channel. This means that no single price system regulates the students' assignments to the undersubscribed schools. Furthermore, a spill-over from the oversubscribed schools accounts for assignment of some students to these schools. Consequently, competitive markets do not extend to them.

Finally, Part (i) asserts ex post Pareto optimality for CADA. At first glance, this feature may be a little surprising in light of the fact that different priority lists are used by different schools. As is clear from DA-MTB, this feature is susceptible to ex post inefficiency. This is not the case, however, with the equilibrium allocation of CADA. To see this, observe first that those students assigned to oversubscribed schools strictly prefer them to any undersubscribed school (or else they should have secured assignment to the latter school by choosing it for their target). This implies that no Pareto improvement can be achieved by reassigning those assigned to oversubscribed schools to undersubscribed schools. In fact, Pareto improvement can only come from reassigning students within oversubscribed schools or from reassinging students within undersubscribed schools. Part (ii) assures, however, that the former is not possible. The latter is not possible since the logic of STB applies to undersubscribed schools.

The characterization of Theorem **??** is tight in the sense that there is generally no bigger set that includes all oversubscribed schools *and some undersubscribed school* that supports ex ante Pareto efficiency.[14]

Theorem **??** refers to an endogenous property of an equilibrium, namely the set of over/under-subscribed schools. We provide a sufficient condition for this property. Let

$$S^* := \{i \in S | m_i \geq 1\}$$

be the set of*popular* schools which each cannot accommodate all the students who prefer them the most. Note that a most popular school must be popular, i.e., $S^{**} \subset S^*$.

It is easy to see that every school in S^* must be oversubscribed in equilibrium. Suppose to the contrary that school $i \in S^*$ is undersubscribed. Then, by Lemma **??**-(ii), every student with $\mathbf{v}$ with $\pi_1(\mathbf{v}) = i$ must subscribe to i, a contradiction. Since each school in S^* is oversubscribed, the next result follows from Theorem **??**.

Corollary 2. *Any equilibrium allocation of CADA is EAPO within the set S^* of popular schools.*

Corollary **??** provides a sufficient condition for a school to be oversubscribed. But it is quite possible that a non-popular school can be oversubscribed in equilibrium. In particular, if all students have the same ordinal preferences, then $|S^*| = 1$, so Corollary **??** has no bite. Yet, the set of oversubscribed schools can be much bigger than S^* even in this case. We can provide some insight into this question, by introducing more structure into the preferences.

Suppose all students have the uniform ordinal preferences, with the schools indexed by the uniform ranking. Letting $\mathcal{V}^U := \{\mathbf{v} \in \mathcal{V} | v_1 > ... > v_n\}$, the students will have the same ordinal preferences if $\mu(\mathcal{V}^U) = \mu(\mathcal{V})$. Define

$$\mathcal{V}_2^U := \left\{\mathbf{v} \in \mathcal{V}^U \middle| \frac{\sum_{i=1}^n v_i}{n} < v_2\right\}.$$

[14]To see this, suppose there are four schools, $S = \{1, 2, 3, 4\}$, and four types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4\}$, with $\mu(\mathbf{v}^1) = \frac{3-\varepsilon}{2}$, $\mu(\mathbf{v}^2) = \frac{1+\varepsilon}{2}$, $\mu(\mathbf{v}^3) = \frac{3-\varepsilon}{2}$, and $\mu(\mathbf{v}^4) = \frac{1+\varepsilon}{2}$ where ε is a small number.

	v_j^1	v_j^2	v_j^3	v_j^4
$j = 1$	10	10	20	20
$j = 2$	3	5	9	8
$j = 3$	1	4	8	1
$j = 4$	0	0	0	0

In this case, type 1 and 3 students subscribe to school 1, and type 2 and 4 students subscribe to school 2. More specifically, the allocation ϕ^* has $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^3) = (\frac{1}{3-\varepsilon}, 0, \frac{2-\varepsilon}{2(3-\varepsilon)}, \frac{2-\varepsilon}{2(3-\varepsilon)})$ and $\phi^*(\mathbf{v}^2) = \phi^*(\mathbf{v}^4) = (0, \frac{1}{1+\varepsilon}, \frac{\varepsilon}{2(1+\varepsilon)}, \frac{\varepsilon}{2(1+\varepsilon)})$. Although schools 1 and 2 are oversubscribed, this allocation is not EAPO within $\{1, 2, 3\}$ since type 1 students can trade probability shares of school 1 and 3 in exchange for probability share at 2, with type 1 students. The allocation is not EAPO within $\{1, 2, 4\}$ either, since type 3 students can trade probability shares of school 1 and 4 in exchange for probability share at 2, with type 4 students. Therefore $\{1, 2\}$ is the largest set of schools that support Pareto efficiency.

Lemma 4. *Assume $\mu(\mathcal{V}^U) = \mu(\mathcal{V})$, then at least two schools are oversubscribed in the CADA equilibrium if $\mu(\mathcal{V}_2^U) \geq 1$.*

Full Pareto optimality may be achieved in some cases.

Corollary 3. *The equilibrium allocation of CADA is EAPO if (i) all but one schools are popular, or if (ii) $n = 3$ and all students have the same ordinal preferences and $\mu(\mathcal{V}_2^U) \geq 1$ holds.*

4.1 Comparison of Procedures

A three-way comparison emerges from the preceding analysis. It provides a formal sense in which the CADA yields a better outcome than DA-STB, which in turn yields a better outcome than DA-MTB. In particular, if the allocation from DA-MTB is Pareto optimal within $K \subset S$, then so is the allocation from DA-STB, although the converse does not hold; and if the allocation from DA-STB is Pareto optimal within $K' \subset S$, then so is the allocation from DA-STB, although the converse does not hold.

Specifically, between the two DA algorithms, DA-STB assigns the seats of the most popular school to those who prefer it the most, whereas this never happens with any school under DA-MTB. Further, the allocation arising from DA-STB is Pareto optimal within any two schools, whereas the allocation from DA-MTB generically fails to be Pareto optimal within two schools unless they contain a worst school.

Meanwhile, the CADA allocation is Pareto optimal within a strictly bigger set of schools than the allocations from DA algorithms, if there are more than two popular schools. The following examples illustrate comparisons further.

Example 1. *There are three schools, $S = \{1, 2, 3\}$, and three types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$, each with $\mu(\mathbf{v}^i) = 1$.*

	v_j^1	v_j^2	v_j^3
$j=1$	5	4	1
$j=2$	1	2	5
$j=3$	0	0	0

Notice that $S^{**} = \{1\}$ and $S^* = \{1, 2\}$. It follows from Corollary **??** *that the allocation from CADA is Pareto optimal. More specifically, the equilibrium allocation is $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^2) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\phi^*(\mathbf{v}^3) = (0, 1, 0)$.*

The allocation from DA-STB is Pareto optimal within any pair of two schools: $\phi^S(\mathbf{v}^1) = \phi^S(\mathbf{v}^2) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ and $\phi^S(\mathbf{v}^3) = (0, \frac{2}{3}, \frac{1}{3})$.[15] *This allocation is not Pareto optimal since student 1 can trade probability shares of schools 1 and 3 in exchange for probability share at school 2, with student 2.*

[15] Assuming that each student has a single uniform draw from $[0, 3]$, the cutoff for school 1 is $c_1 = 1.5$, the cutoff for school 2 is $c_2 = 2$, and the one for school 3 is 3.

The allocation from DA-MTB is $\phi^M(\mathbf{v}^1) = \phi^M(\mathbf{v}^2) \approx (0.392, 0.274, 0.333)$ and $\phi^M(\mathbf{v}^3) \approx (0.215, 0.451, 0.333)$.[16] *This is not Pareto optimal within $\{1, 2\}$.*

Example 2. *There are three schools, $S = \{1, 2, 3\}$, and two types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2\}$, with $\mu(\mathbf{v}^1) = 2$ and $\mu(\mathbf{v}^2) = 1$.*

	v_j^1	v_j^2
$j = 1$	3	3
$j = 2$	1	2
$j = 3$	0	0

In this case, $S^* = S^{**} = \{1\}$. Yet, it follows from Corollary **??** *that the allocation from CADA is Pareto optimal. More specifically, the allocation ϕ^* has $\phi^*(\mathbf{v}^1) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\phi^*(\mathbf{v}^2) = (0, 1, 0)$.*

DA-STB and DA-MTB entail the same allocation $\phi^{DA}(\mathbf{v}^1) = \phi^{DA}(\mathbf{v}^2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is Pareto optimal within any pair of two schools. (The result of Theorem **??***-(iii) does not hold for DA-MTB because the full-support assumption does not hold here.) This allocation is not Pareto optimal since type 1 students can trade probability shares of school 1 and 3 in exchange for probability share at 2, with type 2 students.*

Example 3. *There are three schools, $S = \{1, 2, 3\}$, and two types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2\}$, each with $\mu(\mathbf{v}^1) = 2$ and $\mu(\mathbf{v}^2) = 1$.*

	v_j^1	v_j^2
$j = 1$	10	10
$j = 2$	1	2
$j = 3$	0	0

In this example, the allocation arising from CADA is not Pareto optimal. All students subscribe to school 1 in equilibrium, so the allocation ϕ^* is $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, just as with DA-STB and DA-MTB.

In fact, if all students have the uniform ordinal preferences (i.e., $\mu(\mathcal{V}) = \mu(\mathcal{V}^U)$), then we can show that the CADA (weakly) Pareto-dominates the DA.

Theorem 6. *Suppose all students have the same ordinal preferences. The equilibrium allocation of CADA (weakly) Pareto dominates the allocation arising from DA with any random tie-breaking rule. If $\mu(\mathcal{V}_2^U) > 0$, then there exists a positive measure of students who are strictly better off from the CADA algorithm.*

5 Simulations

The theoretical results in the previous sections do not speak to the magnitude of efficiency gains or loses in each mechanism. In this section, we numerically investigate these questions via simulations, which also help highlight the sources of efficiency gains and loses.

[16] Again, assuming that each student has a uniform draw from $[0, 3]$ for each school separately, the cutoff for school 1 is $c_1 = \frac{5-\sqrt{7}}{2} \approx 1.177$, the cutoff for school 2 is $c_2 \approx 1.354$, and the one for school 3 is 3.

In our numerical model, we have 5 schools each with 20 seats and 100 students. Student i's vNM value for school j, v_{ij}, is given by

$$v_{ij} = \alpha u_j + (1 - \alpha) u_{ij}$$

where $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$, u_j are common across students and u_{ij} is specific to student i and school j. We will say that correlation among preferences increases as α increases. For each α, we draw $\{u_j\}$ and $\{u_{ij}\}$ uniformly randomly and independently to construct student preferences. Given a cardinal utility profile, we simulate DA-STB and DA-MTB, compute a complete information Nash equilibrium of CADA and the resulting CADA outcome. We repeat that 100 times by drawing a new set of vNM utility values for each α. In addition, we solve for a first best solution, which is a utilitarian maximum for each set of vNM utility values. We compute welfare in an experiment as the average of the students' expected utilities in that experiment.[17]

In Figure 1, we compare the three mechanisms to the first best solution. We plot the welfare of each mechanism as the percentage of the welfare at the first best solution. Accordingly, CADA outperforms DA-STB, which outperforms DA-MTB at every value of α. The difference in performance among the mechanisms is exemplified as α gets larger. All three mechanisms perform almost equally well and produce about 90% of the first best welfare when $\alpha = 0$. At $\alpha = 0.9$, CADA achieves 43.5% of the first best welfare, whereas DA-STB achieves 13.7%. A closer look at these extreme cases provides the first insight for the pattern in Figure 1. In particular, when $\alpha = 0$, there is little similarity among preferences and hence about equal number of students rank each school as first choice. As a result, all three mechanisms are likely to assign almost all students to their first choices and all three perform almost equally well when $\alpha = 0$. At the other extreme of $\alpha = 0.9$, almost all students have the same school as first choice. In this case, assigning the right group of students to that school, i.e. the ones with high vNM value for it, becomes crucial for achieving higher welfare. By incorporating students' vNM values in their decision making, CADA outperforms DASTB significantly at $\alpha = 0.9$. Intuitively, if a student's vNM value for a school increases, the likelihood of the student selecting that school as target in an equilibrium of CADA, therefore the likelihood of her getting that school, are expected to increase. This feature of CADA contributes to welfare gains.

Figure 2 gives further insight about the workings of the mechanisms. It shows the percentage of students getting their first choices under each scheme. First, DA-MTB assigns significantly smaller numbers to first choices. This is due to the artificial stability constraints created by multiple tie breaking, which also explains the bigger welfare loss associated with DAMTB. The patterns for CADA and DA-STB are more revealing. In particular, both assign almost the same number of students to their first choices for each value of α. So, whereas the welfare differential in DAMTB can be explained by the decrease in the number of students getting their first choices, the difference among the other two can be explained partly by to which group of students the three mechanisms are assigning first choices more frequently.

[17] See Appendix for a detailed explanation for the simulations and the computation of the numbers for the figures.

In fact, Figure 3 confirms that intuition. This figure shows the the ratio of the mean utility of those who get their k-th choice under CADA to the mean utility of those who get their k-th choice under DA-STB at the realized matchings, for $k = 1, 2, 3$. Accordingly, those who get their k-th choice achieve a higher utility under CADA than under DASTB for each $k = 1, 2, 3$. Note the much more emphasized difference for the receivers of second choice. As one's vNM for her second choice increases, that is, as her second choice becomes a close alternative to her first choice, she is more likely to select her second choice as target in order to avoid stiff competition at her first choice. As a result, she is more likely to get her second choice, therefore those who get their second choices are more likely to achieve higher utility under CADA. This self-selection is demonstrated in Figure 3. The same is true for third choices as well, and more so as α gets larger, which is also demonstrated in the figure.

Figure 4 shows that the number of oversubscribed schools is larger on average than the number of popular schools. Note that the average number of oversubscribed schools is larger than 2 at all values of α. Recalling our Theorems 2 and 5, DASTB is never EAPO within a set of more than 2 schools, whereas CADA is EAPO within the set of oversubscribed schools. Then Figure 4 shows a greater scope of efficiency with CADA, another main source of efficiency gain in CADA in comparison to DASTB.

The equilibrium behavior of students in Figure 5 shows that more students pick their lower ranked schools as target as α gets closer to 1. This monotone pattern in behavior can be explained by the extent of competition over schools. As α, therefore similarity among preferences, increases, more people have the same school as first choice. Therefore competition for one's first choice becomes more intense. This gives students incentive to compete for their second, third and even fourth choices by selecting them as their target school in CADA.[18] This also explains the widening gap between the number of popular schools and the number of oversubscribed schools as α goes to 1 in Figure 4. Note that there exist some realizations of the vNM values with positive probability, under which some students pick their forth choices in equilibrium when α is large.

Next, we numerically investigate CADA when students have priorities at schools. To this end, we introduce schools priorities as follows: Each school has two priority classes, high priority and low priority. For each preference profile above, we assume that 50 students have high priority in their first choice and low priority in their other choices, 30 students have high priority in their second choice and low priority in their other choices, and 20 students have high priority in their third choice and low priority in their other choices.[19] For easier reference, if a student has high priority at a school, we refer that school as that student's neighborhood school. We compute a complete information Nash equilibrium of CADA. Furthermore, we simulate DASTB. In this case, however, priorities introduce inefficiency, that is the matching produced by DASTB or CADA does not need to be student-optimal. Therefore, after computing a DASTB outcome, we find a student optimal stable matching that Pareto dominates the outcome of DASTB via Erdil and Ergin's (forthcoming) stable improvement cycles algorithm, which we refer as the DASTB+SIC outcome.

[18] Note that it is never optimal to select the fifth(last) choice as target.

[19] This assumption is in line with empirical observation in Boston.

In Figure 6, we compare CADA, DASTB and DASTB+SIC to the first best solution which maximizes the sum of utilities by ignoring priorities. We plot the welfare of each mechanism as the percentage of the welfare at the first best solution with mean-adjusted vNM values. Accordingly, CADA outperforms DA-STB for all values of α. DASTB+SIC outperforms CADA up to $\alpha = 0.3$. However, note that this comes at the expense of incentives, as there is no student-optimal and strategy-proof stable mechanism (Erdil and Ergin, forthcoming). When there is more similarity among preferences, that is $\alpha = 0.4$, CADA catches up with DASTB+SIC and outperforms it as α gets bigger, that is similarity among preferences increases. In fact, DASTB+SIC loses its advantage over DASTB as α gets bigger. This is due to the fact that the extent of inefficiency associated with DASTB diminishes as α goes to one. In particular, in the extreme case of $\alpha = 1$, every matching is student optimal so that the stable improvement cycles algorithm has no bite. In contrast, CADA allocates schools more efficiently than DASTB+SIC does for higher values of α.

Figure 7 shows the percentage of students getting their first choices. Whereas CADA always outperforms DASTB, DASTB+SIC assigns more students to their first choices than CADA does for low values of α. The difference vanishes for large values of α. The equilibrium behavior of students under CADA is similar, more and more students pick their lower ranked schools as their target as α gets bigger.[20] However, Figure 8 shows that more students utilize their target choice for their neighborhood schools. The intuition behind this result is subtle. As more and more students pick their lower ranked choices as their target in equilibrium, it becomes tougher to compete for lower ranked schools especially for students who do not have high priority in their lower ranked schools. In turn, those students pick their first choice more frequently, which is their neighborhood school. However, this increases the competition at first choice schools for students whose first choices are not their neighborhood schools. In turn, they pick their lower ranked neighborhood schools as their target. In equilibrium, more students pick their neighborhood schools as their target for larger values of α.

In summary, some similarity among preferences is expected in real-life school choice programs. However, at those instances student optimality, therefore DASTB+SIC, has little bite in improving ex ante efficiency. At those instances, CADA allocates schools more efficiently from an ex ante point of view even though its outcome may be inefficient ex post. CADA achieves this efficiency gain without harming student incentives, whereas ex post student optimality necessarily implies the loss of strategy-proofness.

6 Discussion

6.1 Enriching the Auxiliary Message

The auxiliary message can be expanded to include more than one school, perhaps at the expense of some practicality. In general, the auxiliary message can include a rank order of schools up to

[20] A graph is available from the authors up on request.

$k \leq n$, with the tie broken in the lexicographic fashion according to this rank order: A student is reordered to be ahead of another one at the priority list of school $i \in S$ if and only if the former ranks it higher than the latter in the auxiliary message. We call the associated CADA a ***CADA of degree*** k.

It is worth noting that the CADA of degree n coincides with the Boston mechanism if the schools have no priorities and if all students have the same ordinal preferences. Such an enriching of the auxiliary message does not alter the qualitative features of CADA. In particular, an argument analogous to that of Theorem **??** applies to CADA of any degree, which has a rather surprising implication:

Theorem 7. *If all students have the same ordinal preferences and the schools have no priorities, then the Boston mechanism weakly Pareto dominates the DA algorithm. If $\mu(\mathcal{V}_2^U) > 0$, then there exists a positive measure of students who are strictly better off from the Boston mechanism than from a DA algorithm with any random tie-breaking procedure.*

Expanding the auxiliary message may complicate the deliberation on the part of students and may be practically cumbersome. The beauty of CADA is that the auxiliary message can be kept as simple as practically manageable, if necessary, to $k = 1$ as has been assumed before. What are the benefits from adding more schools in the message? Some observations are easy to make. First, enriching the message does not generally guarantee full Pareto efficiency. Consider Example **??** again. Allowing the students to include the second message, or even a third message, does not make any difference: All students will pick school 1 as their first target and school 2 as their second target, and the precisely the same allocation will arise in equilibrium (which also coincides with one arising from DA-STB). The enriching of message can have a second-order effect, though. The first example illustrates the benefit side.

Example 4. (More is better) *There are 4 schools, $S = \{1, 2, 3, 4\}$, and two types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2\}$, each with $\mu(\mathbf{v}^1) = 3$ and $\mu(\mathbf{v}^2) = 1$.*

	v_j^1	v_j^2
$j = 1$	20	20
$j = 2$	4	3
$j = 3$	1	2
$j = 4$	0	0

With CADA of degree 1, all students subscribe to school 1, so the allocation is completely random with $\phi^*(\mathbf{v}^j) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $j = 1, 2$. With CADA of degree 2, all students pick school 1 as their first target; but type 1 students pick school 2 as their second target whereas type 2 students pick school 3 as their second target. Consequently, the allocation becomes $\phi^{**}(\mathbf{v}^1) = (\frac{1}{4}, \frac{1}{3}, \frac{1}{12}, \frac{1}{3})$ and $\phi^{**}(\mathbf{v}^2) = (\frac{1}{4}, 0, \frac{3}{4}, 0)$. This allocation ϕ^{**} Pareto dominates ϕ^*, although the former is not Pareto optimal.

A richer message need not be always better. A richer message space generates more opportunities for a student to self select at different tiers of schools. But the alternative opportunities may work as substitutes and militate each other. For instance, an opportunity to self select at a lower tier of schools may reduce a student's incentive to self select at a higher tier of schools, even though the latter kinds of self selection may be more important from the social welfare perspective. This kind of "crowding out" arises in the next example.

Example 5. (More is worse) *There are 4 schools,* $S = \{1, 2, 3, 4\}$*, and two types of students* $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2\}$*, with* $\mu(\mathbf{v}^1) = 3$ *and* $\mu(\mathbf{v}^2) = 1$*.*

	v_j^1	v_j^2
$j = 1$	12	8
$j = 2$	2	4
$j = 3$	1	3
$j = 4$	0	0

Consider first CADA of degree 1. Here, all type 1 students choose school 1 as their target, and all type 2 students choose school 2 as their target. In other words, the latter type of students self select into the second popular school. The resulting allocation is $\phi^*(\mathbf{v}^1) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$ *and* $\phi^*(\mathbf{v}^2) = (0, 1, 0, 0)$*. The expected utilities are* $EU^1 = 4.33$ *and* $EU^2 = 4$*. In fact, this allocation is Pareto optimal.*

Suppose now CADA of degree 2 is used. In equilibrium, type 1 students choose school 1 and 2 as their first and second targets, respectively. Meanwhile, type 2 students choose school 1 (instead of school 2!) as their first target and school 3 as their second target. Here, the opportunity for type 2 students to self select at a lower tier school (school 3) blunts their incentive to self select at a higher tier school (school 2). The resulting allocation is thus $\phi^{**}(\mathbf{v}^1) = (\frac{1}{4}, \frac{1}{3}, \frac{1}{12}, \frac{1}{3})$ *and* $\phi^{**}(\mathbf{v}^2) = (\frac{1}{4}, 0, \frac{3}{4}, 0)$*, which yield expected utilities of* $\overline{EU}^1 = 3.75$ *and* $\overline{EU}^2 = 4.25$*. This allocation is not Pareto optimal since type* 2 *students can trade probability shares of school* 1 *and* 3 *in exchange for probability share at* 2*, with type* 1 *students.*

Even though ϕ^* *does not Pareto dominate* ϕ^{**}*, the former is Pareto optimal whereas the latter is not. Further, the former is superior to the latter in the Utilitarian sense (recall that students' payoffs are normalized so that they aggregate to the same value for both types): the former gives aggregate utilities of* 17*, the highest possible level, whereas the latter gives* 15.5 *(which is* 0.5 *above the level that would arise from random assignment).*

The last example suggests that the benefit from enriching the message space is not unambiguous. This is a potentially important point. In practice, expanding a message space adds a burden on the parents to be strategically more sophisticated. Hence avoiding such a demand for strategic sophistication is an important quality for a procedure to succeed. This makes the simple CADA (i.e., of degree 1) quite appealing. That this practical benefit may not even involve a welfare sacrifice is reassuring about the simple CADA.

6.2 Strategic Naivety

Since CADA involves some "gaming" aspect, albeit limited to tie-breaking, a natural concern is that not all families may be strategically competent. This concern has arisen in the context of the Boston mechanism. It has been observed that some significant percentage of families have played suboptimal strategies, for instance, wasting their second top choices to schools that are so popular that students can get in those schools by listing them as top choices. Such mistakes may arise because of the lack of knowledge about how the system works or the difficulty with assessing how popular schools are. The same concern may arise with respect to CADA, in that some families may not understand well the role the auxiliary message plays in the system and/or they may not judge accurately how over/undersubscribed various schools will turn out.

It is thus important to investigate how the CADA will perform when some families are not strategically sophisticated. To this end, we consider students who are "naive" in the sense that they always name their most preferred schools for target in the auxiliary message. Naming the most preferred school appears to be a simple, but reasonable, choice when she/he is unsure about the popularity of alternative schools or unclear about the role the auxiliary message plays in the assignment. Such a strategy will indeed be a best response for many situations, particularly if the first choice is distinctively better than the rest of the choices, so it could be a good approximation. We assume that there is a positive measure of students who are naive in this way, and the others know the presence of these students and their behavior, and respond optimally against them. Surprisingly, the presence of naive students do not affect the main welfare results in a qualitative way.

Theorem 8. *In the presence of naive students, the equilibrium allocation of CADA satisfies the following properties:*

(i) The allocation is EPPO, and is thus pairwise EAPO.

(ii) The allocation is EAPO within the set K of oversubscribed schools.

(iii) If every student is naive, then the allocation is EAPO within $K \cup \{l\}$ for any undersubscribed school $l \in J := S \backslash K$.

Theorem **??**-(i) and (ii) are qualitatively the same as the corresponding parts of Theorem **??**, except for Part (iii) of Theorem **??**.[21] Of course, the set of oversubscribed schools need not be the

[21] Suppose there are three schools, $S = \{1, 2, 3\}$, and three types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$, with $\mu(\mathbf{v}^1) = \mu(\mathbf{v}^2) = \mu(\mathbf{v}^3) = 1$.

	v_j^1	v_j^2	v_j^3
$j = 1$	10	10	10
$j = 2$	1	8	9
$j = 3$	0	0	0

Suppose type 1 and 2 students are sophisticated while type 3 students are all naive. In this case, type 1 and 3 students submit school 1, and type 2 students submit school 2 as their target. Then, the resulting ex ante allocation has $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^3) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\phi^*(\mathbf{v}^2) = (0, 1, 0)$. Although schools 1 and 2 are oversubscribed, this allocation same when some fraction of students are naive, so (even) these results do not admit direct comparison between the case of fully rational students and the current case. In particular, Theorem **??**-(iii) does not mean that the Pareto optimal set of schools is larger when all students are naive than when there are no naive students. When every student is naive, the set of oversubscribed schools coincides with the set of popular schools. When no students are naive, however, the former is always weakly larger than the latter and can be strictly larger (Recall Examples 1 and 3).

Nevertheless, the efficiency statement is very similar. In particular, Lemma **??**-(ii) remains valid in the current context, implying that any popular schools must be oversubscribed here as well. Hence, the same conclusion as Corollary **??** holds.

Corollary 4. *In the presence of naive students, the equilibrium allocation of CADA is Pareto optimal within the set S^* of popular schools.*

We also investigate numerically the impact of naive players on average welfare via simulations. To this end, we assume no priorities at schools and that a certain number of students play naively and select their first choice as their target schools. Other students play their best responses in a complete information Nash equilibrium. We run this simulation with 50 naive students, with 75 naive students and with 100 (all) naive students. Figure 9 reports the average percentage welfare with respect to to the first best with zero, 50, 75 and 100 (all) naive students. The welfare patterns are similar. CADA with any number of naive players outperforms DASTB, which ouperforms DAMTB. A bigger number of strategic players yields a more efficient outcome. It is worth to note that CADA continues to outperform DA-STB even when all players are naive.

6.3 CADA with "Safety Valve"

The preceding subsection has seen that the main welfare property of CADA remains valid even when an arbitrary proportion of the student population behaves naively. This does not mean, however, that naive students are not disadvantaged by the others who may make strategic use of the message. It thus makes sense to provide an extra safeguard to those who may feel unsure about how to play the CADA game. This can be done by augmenting the message space to allow for an "exit option" and to treat those who invoke such an option *as if* they are participating in the standard DA algorithm. Specifically, the CADA can be modified as follows.

- All students submit ordinal preferences and auxiliary messages. In the auxiliary message, a student can name a target school or say "Opt Out."
- Random ordering of students are generated according to the standard method (e.g., STB). Then, run DA-STB using this priority list. (If a school has inherent priorities, the random list is used only to break a tie within the same priority class.) Assign those who have picked "Opt Out" according to this procedure.

is not EAPO since it will be Pareto improving for type 2 students to trade probability share of school 2 in exchange for probability shares at schools 1 and 3, with type 3 (naive) students. Therefore, Theorem **??**-(iii) does not extend to the case in which we have *both* sophisticated and naive students.

- Assign the remaining students to the remaining seats, using the CADA algorithm. Specifically, construct the choice-augmented priority list for each school, as described before (using two random lists). Then, the assignment is made via the DA algorithm using the choice-augmented priority.

Clearly, this modified algorithm gives each student the option of achieving precisely the same lottery of assignments as she receives from the DA-STB. But she can choose to send an active signal and do better.

Theorem 9. *The CADA with the safety option makes every student (weakly) better off than he/she is from the DA-STB.*

6.4 Dynamic Implementation

As noted, the welfare benefit of CADA originates from the competitive markets it induces. Unlike the usual markets where there are explicit prices, however, in the CADA-generated markets, students' beliefs about the relative popularity of schools act as the prices. Hence, for the CADA to have the desirable welfare benefit, the students beliefs must be reasonably accurate. In practice, the students' preferences tend to reflect the reputations that schools have developed; thus, as long as the school reputations are stable, they can serve as reasonably good proxies for the prices. Nevertheless, the students may not share the same beliefs and the beliefs may not be accurate, in which case CADA procedure will not implement the CADA equilibrium precisely.

The CADA equilibrium can be implemented more precisely by making the (shadow) prices more explicit and by facilitating students' ability to respond to them. Suppose, after submitting their ordinal preferences (which is an once and for all decision), the students can be asked to submit their auxiliary message in a dynamic fashion.

In Round 1, the students submit the names of their target schools. At the end of Round 1, the students are allowed to see the population distribution of target school choices.[22] In Round $m \geq 2$, the students are allowed to change their auxiliary messages. If the number of students who have changed their auxiliary messages from the previous round is less than some (pre-specified) threshold, then their choices in Round m become final, and CADA is run, just as before, to produce a matching. If the number exceeds the threshold, then the population distribution of choices is announced, and they move on to Round $m + 1$.

Under this dynamic mechanism, a student's choice matters only when most of the other students do not alter their choices. It is thus optimal for students to simply best respond to the announced distribution of population choices in the previous round. Clearly, the resulting best-response dynamics will lead to a Nash equilibrium (i.e., a CADA equilibrium discussed earlier), whenever the process converges. Although our dynamic process likely converge in practice, activity rules can be added to facilitate the convergence. For instance, limit cycles can be eliminated by

[22] Alternatively, the clearinghouse may compute and display the probabilities of assignment to different schools that would arise (from the CADA) from each choice, assuming that no other student alters her auxiliary message.

preventing some small fraction of randomly selected students from returning back to their original choices after deviating from them once or twice.

7 Conclusion

In this paper, we propose a new deferred acceptance procedure in which students are allowed, via signaling of their preferences, to influence how they are treated in a tie for a school. This new procedure, choice-augmented DA algorithm (CADA), makes the most of two existing procedures, the Gale-Shapely's deferred acceptance algorithm (DA) and the Boston mechanism. While the DA achieves the strategyproofness, an important property in the design of school choice programs, it limits students' abilities to communicate their preference intensities, which entails an ex ante inefficient allocation when schools are indifferent among students with the same ordinal preferences. The Boston mechanism, on the other hand, is responsive to the agents' cardinal preferences and may achieve more efficient allocation than the DA, but fails to satisfy strategyproofness. We show that, by allowing students to influence tie-breaking via additional communication, CADA implements a more efficient ex ante allocation than the standard DA algorithms, without sacrificing the strategyproofness of ordinal preferences.

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Appendix A: Extensions of Algorithms to the Continuous Environment

We define the alternative DA procedures here.

Ordinal preferences. In any DA algorithm, every student submits a ranking of schools. Formally, students' ordinal preferences are represented by a measurable function $P : \Theta \to \Pi^n$, where $P(\mathbf{v}, \omega) \in \Pi^n$ is an ordered list of n schools (ordered not necessarily according to true preferences). Since the ω is unobserved by the students (at least at the time of submitting the ordinal preferences), we require that $P(\mathbf{v}, \omega) = P(\mathbf{v}', \omega')$ whenever $\mathbf{v} = \mathbf{v}'$. We say a DA algorithm is *ordinally strategy-proof* if it is a (weak) dominant strategy for each student with $\mathbf{v}$ to choose $P(\mathbf{v}, \omega) = \pi^n(\mathbf{v})$.

Definition of DA algorithms: Given ordinal preferences P and a tie-breaker $\mathcal{F} = \{F_i : i \in S\}$, a DA algorithm is defined as follows. First, we define a measurable function Ch_{F_i} over subsets of Θ as the set of best ranked students for school $i \in S$ according to F_i from a given set up to the capacity. Formally, for any measurable $X \subset \Theta$, let

$$Ch_{F_i}(X) := \sup\{Y \subset X | \eta(Y) \leq 1, F_i(\theta) < F_i(\theta'), \forall \theta \in Y, \theta' \in X \backslash Y\}$$

denote the set of students chosen from X such that the set does not exceed the capacity and that the chosen students have a higher priority than those not chosen.

Next, we define the $DA_{\mathcal{F}}$ (deferred acceptance) mapping. Consider first a mapping $Q : \Theta \to \Pi$, where $Q(\theta)$ is an ordered list of any $k \leq n$ schools. (Recall $P(\theta)$ is a special case involving the full set of schools.) The DA mapping, $Q' = DA_{\mathcal{F}}(Q) \in \Pi$ is determined as follows. Every student with θ applies to her most preferred school in $Q(\theta)$. Every school i (tentatively) admits from its applicants in the order of F_i. If all of its seats are assigned, it rejects the remaining applicants. If a student θ is rejected by i, $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting i in $Q(\theta)$. If a student θ is not rejected, then $Q'(\theta) = Q(\theta)$. More formally, let $T_i(Q) = \{\theta \in \Theta : i$ is ranked first in $Q(\theta)\}$ be the set of students that rank i as first choice. Note that $T_i(Q)$ is measurable. Then each school i admits students in $Ch_{F_i}(T_i(Q))$ and rejects students in $T_i(Q) \backslash Ch_{F_i}(T_i(Q))$. If $\theta \in T_i(Q) \backslash Ch_{F_i}(T_i(Q))$ for some $i \in S$, then $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting i from the top of $Q(\theta)$; otherwise $Q'(\theta) = Q(\theta)$. Since Q is a measurable function, Q' is also measurable.

Repeated application of the $DA_{\mathcal{F}}$ mapping gives us the DA algorithm. That is, given a problem $(P, \mathcal{F})$, let $Q^0 = P$ and define $Q^t = DA_{\mathcal{F}}(Q^{t-1})$ for $t > 0$. Then Q^t converges almost everywhere to some measurable Q^* (Theorem **??** below). The matching can be then found by assigning θ to its top choice of $Q^*(\theta)$. Formally, define a mapping $\psi^{(P,\mathcal{F})} : \Theta \mapsto \Delta$ such that $\psi_i^{(P,\mathcal{F})}(\theta) = 1$ if i is the top choice of $Q^*(\theta)$, and $\psi_i^{(P,\mathcal{F})}(\theta) = 0$ otherwise. Since the schools' capacities are respected in each round and also in the limit, the mapping must be an ex post allocation.

We present two main results:

Well-definedness of the Procedure. The existence of $\psi^{(P,\mathcal{F})}$ follows from the next theorem.

Theorem 10. *For every $(P, \mathcal{F})$, $DA^t_{\mathcal{F}}(P)$ converges almost everywhere to some measurable $Q^* : \Theta \to \Pi$.*

Proof. Define the set of rejected students as $R^t = \{\theta : \theta \in T_i(Q^t) \backslash Ch_{F_i}(T_i(Q^t))$ for some $i \in S\}$. Then $\eta(R^t)$ goes to zero as t goes to infinity. Otherwise, if $\eta(R^t) \geq \kappa > 0$ for all t, all the schools in every student's preference would be deleted in finite time because of finiteness of the number of schools, which in turn would imply that $\eta(R^t)$ goes to zero, a contradiction. Therefore, $DA^t_{\mathcal{F}}(P)$ converges almost everywhere to some Q^*. Since every $Q^t = DA^t_{\mathcal{F}}(P)$ is measurable, Q^* is also measurable. ▌

Ordinal Strategy-proofness. Fix arbitrary ordinal preferences P. Let $P_{-\mathbf{v}} : \mathcal{V} \backslash \{\mathbf{v}\} \to \Pi^n$ denote the ordinal preferences of all students but $\mathbf{v}$ determined by P. Recall that $\pi^n(\mathbf{v}) \in \Pi^n$ represents the truthful ordinal preference induced by $\mathbf{v}$, that is $\pi^n(\mathbf{v})$ lists i before j if and only if $v_i > v_j$. To simplify the notation, let $\psi^P := \psi^{(P,\mathcal{F})}$, with $\mathcal{F}$ suppressed, and let $\psi^* := \psi^{(\pi^n, P_{-[\cdot]}, \mathcal{F})}$ denote the matching outcome for any given type when it submits its ordinal preferences truthfully and the others report P. When students report P, a student with type $\mathbf{v}$ receives expected utility of

$$\mathbb{E}_\omega \left[v \cdot \psi^P(\mathbf{v}, \omega) \right].$$

Theorem 11. *For every $(P, \mathcal{F})$, it is a (weak) dominant strategy for every student to submit her ordinal preferences truthfully to DA, that is, for all $\mathbf{v} \in \mathcal{V}$, P,*

$$\mathbb{E}_\omega \left[\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega) \right] \geq \mathbb{E}_\omega \left[\mathbf{v} \cdot \psi^P(\mathbf{v}, \omega) \right].$$

Proof. It suffices to show that, for all $\theta = (\mathbf{v}, \omega)$,

$$\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega) \geq \mathbf{v} \cdot \psi^P(\mathbf{v}, \omega).$$

Suppose to the contrary that

$$\mathbf{v} \cdot \psi^*(\mathbf{v}, \omega) < \mathbf{v} \cdot \psi^P(\mathbf{v}, \omega), \tag{2}$$

for some $\theta = (\mathbf{v}, \omega)$ and P. We show that there exists a finite many-to-one matching problem for which a DA algorithm fails strategy-proofness, which will then constitute a contradiction to the standard strategy-proofness result (Dubins and Friedman, 1981; Roth, 1982).

To begin, fix any $K \in \mathbb{N}_+$, and construct a discretization of (P, F) for θ as follows: For every $\mathbf{z} = (z_1, ..., z_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ where $z_i, y_j \in \{0, ..., K\}$, consider a set

$$\Theta_{\mathbf{z},\mathbf{y}} = \left\{ (\tilde{\mathbf{v}}, \tilde{\omega}) \in \Theta : \frac{z_i}{K} \leq \tilde{v}_i \leq \frac{z_i + 1}{K}, \frac{ny_j}{K} \leq \tilde{\omega}_j \leq \frac{n(y_j + 1)}{K}, i, j \in S \right\}.$$

Let $\eta_{K,\min} = \min_{\Theta_{\mathbf{z},\mathbf{y}}} \eta(\Theta_{\mathbf{z},\mathbf{y}})$, and let $\#\Theta_{\mathbf{z},\mathbf{y}}$ be the integer part of $\frac{\eta(\Theta_{\mathbf{z},\mathbf{y}})}{\eta_{K,\min}}$.

Pick $\#\Theta_{\mathbf{z},\mathbf{y}}$ students in total from every set $\Theta_{\mathbf{z},\mathbf{y}}$ at random without repetition. Let $\{\theta^l\}$ denote the set of students that are picked. If $\frac{|\{\theta^l\}|}{n}$ is not an integer, pick additional students from the larger sets until obtaining an integer $\frac{|\{\theta^l\}|}{n}$. Note that the number of additional students to be picked this way is less than n and n is fixed, therefore this will be negligible in the limit as K goes to infinity. Now consider the problem in which the set $\{\theta^l\}$ of students are to be assigned to a set S of schools each with capacity $\frac{|\{\theta^l\}|}{n}$. Each student $\theta^l = (\mathbf{v}^l, \omega^l)$'s strict ordinal preference is given by $P(\theta^l)$. The schools' strict preferences are given by $\mathcal{F}$. Denote this problem by $(\{\theta^l\}, S, P, \mathcal{F})_K$, and the associated ex post allocation ψ_K^P. As K goes to infinity, $(\{\theta^l\}, S, P, \mathcal{F})_K$ approximate $(\Theta, S, P, \mathcal{F})$ arbitrarily closely. Hence, $\psi_K^P \to_{a.e} \psi^P$ and $\psi_K^{\pi^n(\mathbf{v}),P_{-\mathbf{v}}} \to_{a.e.} \psi^*$ as $K \to \infty$. Hence, if (**??**) holds, then there exists K such that

$$\mathbf{v} \cdot \psi_K^*(\mathbf{v}, \omega) < \mathbf{v} \cdot \psi_K^P(\mathbf{v}, \omega).$$

This contradicts the fact that, in every finite problem, submitting true preferences to the student-proposing deferred acceptance mechanism is a dominant strategy for every student (Dubins and Friedman, 1981; Roth, 1982). ▌

Appendix B: Proofs of the Main Results

Proof of Lemma ??. Part (i) follows trivially since a within-K reallocation of any allocation is its within-K' reallocation for $K' \supset K$. To prove (ii), suppose an ex post allocation ψ is not EPPO within K. Then, there must be an within-K ex post allocation $\tilde{\psi}$ of ψ that Pareto dominates ψ. Clearly, the ex ante allocation $\tilde{\phi}(\cdot) = \int \tilde{\psi}(\cdot, \omega)\xi(d\omega)$ is a within-K ex ante reallocation of $\phi(\cdot) = \int \psi(\cdot, \omega)\xi(d\omega)$, and it must Pareto dominate ϕ, proving that ϕ cannot be EAPO within K.

To prove part (iii), suppose an ex post allocation ψ induces an ex ante allocation ϕ that is not EAPO within $\{i, j\}$. Let $\tilde{\phi}$ be the within-$\{i, j\}$ reallocation of ϕ that Pareto dominates it. Let $\mathcal{V}_i := \{\mathbf{v} \in \mathcal{V} | \tilde{\phi}_i(\mathbf{v}) > \phi_i(\mathbf{v})\}$ and $\mathcal{V}_j := \{\mathbf{v} \in \mathcal{V} | \tilde{\phi}_i(\mathbf{v}) < \phi_i(\mathbf{v})\}$. Almost all $\mathbf{v} \in \mathcal{V}_i$ must have $v_i > v_j$ and almost all $\mathbf{v} \in \mathcal{V}_j$ must have $v_i < v_j$. Since $\phi_i(\mathbf{v}) + \phi_j(\mathbf{v}) = \tilde{\phi}_i(\mathbf{v}) + \tilde{\phi}_j(\mathbf{v})$, for each $\mathbf{v} \in \mathcal{V}_i$, $\phi_j(\mathbf{v}) > 0$ and for each $\mathbf{v} \in \mathcal{V}_j$, $\phi_i(\mathbf{v}) > 0$. Consequently, there must be sets, $A \subset \mathcal{V}_i \times \Omega$ and $B \subset \mathcal{V}_j \times \Omega$, both with positive measure, such that the ex post allocation ψ inducing ϕ has $\psi_j(\theta) = 1$ for each $\theta \in A$ and $\psi_i(\theta) = 1$ for each $\theta \in B$. One can then find $A' \subset A$ and $B' \subset B$ with $\eta(A') = \eta(B') > 0$. And, we can construct a new ex post allocation $\tilde{\psi}$ such that $\tilde{\psi}(\theta) = \psi(\theta)$ for all $\theta \in \Theta \backslash (A' \cup B')$ and $\tilde{\psi}_i(\theta) = 1, \tilde{\psi}_{-i}(\theta) = 0$ for $\theta \in A'$, and $\tilde{\psi}_j(\theta) = 1$ and $\tilde{\psi}_{-j}(\theta) = 0$ for $\theta \in B'$. By construction, $\tilde{\psi}$ is within-$\{i, j\}$ ex post reallocation of ψ. Further, it Pareto dominates ψ.

Part (iv) follows from part (i) and part (iii). ▌

Proof of Lemma ??. Consider the first cutoff $\hat{c}^1$. Suppose this is the cutoff for school i. Take any student. If the student's top choice is not i. Then, if she ever gets to school i — meaning she is turned down by schools she lists ahead of i — then it means that her draw $\omega > c^i$, so she will never get into school i. This means that only students whose most preferred school is i can only get assigned to school i. It then follows that $m_i \cdot \frac{\hat{c}_i^1}{n} = 1$, so $\hat{c}_i^1 = \frac{n}{m_i}$. For $\hat{c}^1$ to be the lowest cutoff, we must have $\hat{c}^1 = \frac{n}{\max_{i \in S} m_i}$. Hence, a most popular school has the lowest cutoff. We conclude that $\hat{c}^1$ is uniquely determined by $\mathfrak{m}$ (more precisely, by $\max\{m_i\}$, which is a function of $\mathfrak{m}$). Since $\max_{i\in S} m_i \geq 1$, $\hat{c}^1 \in (1, n]$.

We work recursively to define the rest of the cutoffs. Suppose that cutoffs $\{\hat{c}^j\}$, $j < k$, are uniquely determined by $\mathfrak{m}$ such that $\hat{c}^i \leq \hat{c}^j \leq n$ for all $1 \leq i < j \leq k-1$. Let the cutoff $\hat{c}^j$, $j \leq k-1$, be the cutoff of school $\kappa(j) \in S$, where $\kappa : \{1, ..., k-1\} \mapsto S$ is one-to-one. We now determine $\hat{c}^k$ uniquely as a function of $\mathfrak{m}$ and establish $\hat{c}^k \geq \hat{c}^{k-1}$. Let $S^{k-1} := \{j | j = \kappa(j') \text{ for some } j' \leq k-1\}$ be the associated set. Suppose that the cutoff $\hat{c}^k$ determines the cutoff of school $i \in S \backslash S^{k-1}$. Then, arguing as before, a student who prefers a school $j \in S \backslash (S^{k-1} \cup \{i\})$ to each school in $S^{k-1} \cup \{i\}$ never stands a chance to get in i. (Clearly, κ, S^{k-1} and $\hat{c}^j$ all depend on $\mathfrak{m}$, which we suppress for convenience.)

For any nonempty subset $S' \subset S^{k-1}$, let $\hat{\Pi}(S')$ be the set of all permutations of S'. Let $\chi(S') := \{j \in S' | \kappa^{-1}(j) \geq \kappa^{-1}(j'), \forall j' \in S'\}$ be the school which has the highest index in S^{k-1}, meaning that $\chi(S')$ will be the school that has the largest cutoff among S' (yet still has a lower cutoff than $\hat{c}^k$. Then, for school i to have cutoff $\hat{c}^k$, the cutoff must be $\hat{c}^k = \hat{c}^k_i$, where $\hat{c}^k_i$ satisfies

$$m_i \cdot \frac{\hat{c}^k_i}{n} + \sum_{S' \subset S^{k-1}} \left[\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \left(\frac{\hat{c}^k_i - \hat{c}^{\chi(S')}}{n} \right) \right] = 1,$$

or

$$\hat{c}^k_i = \frac{n + \sum_{S' \subset S^{k-1}} \left[\hat{c}^{\chi(S')} \left(\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \right) \right]}{m_i + \sum_{S' \subset S^{k-1}} \left(\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \right)}.$$

Let $\hat{c}^k := \min\{n, \min_{j \in S \backslash S^{k-1}} \hat{c}^k_j\}$, and $i = \kappa(k) := \arg\min_{j \in S \backslash S^{k-1}} \hat{c}^k_j$. Note that this definition conforms to the case of $k = 1$. We must have

$$m_i \cdot \frac{\hat{c}^k}{n} + \sum_{S' \subset S^{k-1}} \left[\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \left(\frac{\hat{c}^k - \hat{c}^{\chi(S')}}{n} \right) \right] \leq 1, \tag{3}$$

where the inequality holds with equality if $\hat{c}^k < n$. We next show that $\hat{c}^k \geq \hat{c}^{k-1}$. Suppose to the contrary that $\hat{c}^k < \hat{c}^{k-1} \leq n$. We can rewrite (**??**) (with equality) as

$$m_i \cdot \frac{\hat{c}^k}{n} + \sum_{S' \subset S^{k-2}} \left[\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \left(\frac{\hat{c}^k - \hat{c}^{\chi(S')}}{n} \right) \right] +$$
$$\sum_{S' \subset S^{k-1}, S' \not\subset S^{k-2}} \left[\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \left(\frac{\hat{c}^k - \hat{c}^{\chi(S')}}{n} \right) \right] = 1.$$

Hence,

$$m_i \cdot \frac{\hat{c}^k}{n} + \sum_{S' \subset S^{k-2}} \left[\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau,i)} \left(\frac{\hat{c}^k - \hat{c}^{\chi(S')}}{n} \right) \right] \geq 1,$$

from which it follows that $\hat{c}^k_i \leq \hat{c}^k < \hat{c}^{k-1}$, contradicting the definition of $\hat{c}^{k-1}$.

Let $l \in S \backslash S^{n-1}$ be the last school left. We prove $\hat{c}_l^n = n$. Recall

$$\hat{c}_l^n = \frac{n + \sum_{S' \subset S^{n-1}} \left[\hat{c}^{\chi(S')} \left(\sum_{\tau \subset \hat{\Pi}(S')} m(\tau, l) \right) \right]}{m_l + \sum_{S' \subset S^{n-1}} \left(\sum_{\tau \subset \hat{\Pi}(S')} m(\tau, l) \right)}. \tag{4}$$

The denominator of (**??**) measures of all students, so it equals n.[23] The second term in the numerator of (**??**) becomes, when divided by n,

$$\sum_{S' \subset S^{n-1}} \left[\frac{\hat{c}^{\chi(S')}}{n} \left(\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau, l)} \right) \right],$$

which measures all students who are assigned to S^{n-1} and thus equals the sum of all terms on the left side of (**??**) across $k = 1, ..., n-1$. It thus follows that

$$\sum_{S' \subset S^{n-1}} \left[\frac{\hat{c}^{\chi(S')}}{n} \left(\sum_{\tau \subset \hat{\Pi}(S')} m_{(\tau, l)} \right) \right] \leq n - 1. \tag{5}$$

Substituting (**??**) into (**??**) gives

$$\hat{c}_l^n \leq \frac{n + n(n-1)}{n} = n.$$

To prove $\hat{c}_l^n = n$, suppose $\hat{c}_l^n < n$. Then, by monotonicity, $\hat{c}^k \leq \hat{c}^n \leq \hat{c}_l^n < n$, so (**??**) must hold with equality for all $k = 1, ..., n-1$, which means that (**??**) must hold with equality. Therefore, $\hat{c}_l^n = n$, a contradiction. We conclude that $\hat{c}^n = \hat{c}_l^n = n$.

Although it is possible for $\hat{c}^k = \hat{c}^{k+1}$ for some $i = 1, ..., n-1$, it is easy to see that this is not generic. If $\hat{c}^k = \hat{c}^{k+1}$, this means that there are $i \neq j$ such that $\hat{c}_i^k = \hat{c}_j^{k+1}$, which entails a loss of dimension for $\mathfrak{m}$ within $\mathfrak{M}$. Hence, the Lebesque measure of the set of $\mathfrak{m}$'s involving such a restriction is zero. It thus follows that $\hat{c}^i < \hat{c}^j$ if $i < j$ for a generic $\mathfrak{m}$. ▌

Proof of Theorem ??. The proof is a direct application of Theorem 2 of Mas-Colell (1984). ▌

Proof of Lemma ??. Part (i) follows trivially since such a student can name that school as the target and get assigned to it with probability one. To prove part (ii) consider any student of type $\mathbf{v}$, whose values are all distinct. There are μ-a.e. such $\mathbf{v}$. Suppose her most-preferred school $\pi_1(\mathbf{v}) =: i$ is undersubscribed and not a worst school. It is then her best response to pick i as her target, since doing so can guarantee assignment to i for sure, whereas choosing some other school as the target may result in assignment to some other school. Hence, the student must be choosing i as her target in equilibrium.

[23]The denominator consists of measures of all students whose most preferred school is l, and of all the student whose second preferred school is l, and so on and so forth, thus telescoping to the sum of all students.

To prove part (iii), consider any $\mathbf{v}$ (with distinct values), such that $\pi_1(\mathbf{v}) \neq w$. Suppose first $\nu_i^*(\mathbf{v}) > 0$ for some oversubscribed school i. It follows from the above observation that her most preferred school must be an oversubscribed school (not necessarily i). Given the distinct values, she must strictly prefer school i to all undersubscribed schools. Hence, she lists i ahead of all undersubscribed schools in her ordinal list. Whenever she picks i, she will fail to place in any oversubscribed schools other than i that she may list ahead of i, so she will apply to school i with probability one. Suppose next $\nu_j^*(\mathbf{v}) > 0$ for some undersubscribed school j. Then, the student must prefer j to all other undersubscribed schools, so she will apply to j with probability one whenever she fails to place in any oversubscribed school she may list ahead of j in the ordinal list. Whenever she picks j as her target, she is surely rejected by all oversubscribed schools she may list ahead of j, so she will apply to j with probability one. We thus conclude that $\nu^*(\mathbf{v}) = \bar{\nu}^*(\mathbf{v})$ for μ-a.e. $\mathbf{v}$. ▌

Proof of Theorem ??:

Part (i): By definition, the most popular school has the lowest cutoff say $\hat{c}^1$. No student whose most preferred school is not the most popular school will never be assigned to that school, since whenever she fails to get into a more preferred school than the most popular school, she must have a draw $\omega > \hat{c}^1$.

Part (ii): Let ψ^S be the ex post allocation arising from DA-STB. We show there is no ex post allocation that Pareto dominates ψ^S. To this end, list $S = \{i_1, ..., i_n\}$ in the ascending order of the equilibrium cutoffs of the scores (with the schools with the same cutoff listed in an arbitrary order), i.e., i_k has a weakly lower cutoff than i_l if $k < l$. Any student assigned to a school say $i_k \in S$ must have listed that school ahead of every school i_l with $l > k$ (i.e., a weakly higher cutoff) in the ordinal ranking; or else, she could never have been assigned to i_k. Given the ordinal strategyproofness (Theorem **??**), almost all these students must be strictly worse off from being reassigned to any school i_l with $l > k$. Suppose to the contrary that an ex post allocation, say $\tilde{\psi}$, Pareto dominates ψ^S. There must be a school $i_k \in S$ such that a positive measure of θ's assigned to i_k under ψ^S must be assigned to some other school(s) under $\tilde{\psi}$. Let i_{k^*} be such a school with the smallest cutoff index k^*. Then, almost all types reassigned from i_{k^*} must be reassigned to school i_l, with $l > k^*$, under $\tilde{\psi}$. Almost all such student type must be strictly worse off from the reassignment. Hence, $\tilde{\psi}$ cannot Pareto dominate ψ^*, a contradiction.

Part (iii): By Lemma **??**, for a generic $\mathfrak{m}$, the cutoff scores are distinct. Fix any such $\mathfrak{m}$, and let $\hat{c}^1 < ... < \hat{c}^n$ be the cutoff scores. Let $\bar{\kappa} : S \mapsto S$ be a permutation such that $\hat{c}^i = \hat{c}^i_{\bar{\kappa}(i)}$, and let $\bar{\tau} = (\bar{\kappa}(1), ..., \bar{\kappa}(n))$ be the resulting list of schools. Now consider the set of $\mathcal{V}_{\bar{\tau}} := \{\mathbf{v} \in \mathcal{V} | v_{\bar{\kappa}(i)} > v_{\bar{\kappa}(j)} \text{ if } i < j\}$ student types that have the same preference order as $\bar{\tau}$. We first prove that any such student type has positive probability of assigning to every school in DA-STB. This can be shown as follows. By the full support assumption, $m_{\bar{\tau}} = \mu(\mathcal{V}_{\bar{\tau}}) > 0$. For any school $i \in S$, a student type $\mathbf{v} \in \mathcal{V}_{\bar{\tau}}$ will be assigned to that school whenever her draw ω lies in between $\hat{c}^{\bar{\kappa}^{-1}(i)-1}$ and $\hat{c}^{\bar{\kappa}^{-1}(i)}$ (let $\hat{c}^0 \equiv 0$), since her draw will not be low enough to get in any school she prefers but low enough to get in i. Hence, her probability of getting assigned to school i is $\frac{\hat{c}^{\bar{\kappa}^{-1}(i)}-\hat{c}^{\bar{\kappa}^{-1}(i)-1}}{n} > 0$ since the cutoffs are distinct.

Take any $K = \{k_1, k_2, k_3\} \subset S$. We show that the DA-STB allocation ϕ^S fails to be EAPO within K. Consider again the student type $\mathbf{v} \in \mathcal{V}_{\bar{\tau}}$. Without loss, $v_{k_1} > v_{k_2} > v_{k_3}$ for any such type. By the above observation, $\phi^S_{k_i}(\mathbf{v}) > 0$ for $i = 1, 2, 3$. By the full support assumption, there is a Pareto dominating within-K reallocation of ϕ^S among the student types in $\mathcal{V}_{\bar{\tau}}$. Those with a high v_{k_2} relative to (v_{k_1}, v_{k_3}) sell shares at (k_1, k_3) in exchange for an increased share at k_2, with those with a low v_{k_2} relative to (v_{k_1}, v_{k_3}). We thus conclude that there exists no K with $|K| = 3$ such that ϕ^S is EAPO within K. The statement holds then by Lemma **??**-(i) and (iii). ▌

Proof of Theorem ??:

Part (i): For generic $\mathfrak{m}$, there are at least two schools, say i and j, whose cutoffs are strictly below n, so a positive measure of those who prefer either of these schools the most are not assigned to that school. A positive measure of those who prefer i the most but not assigned to i have $k \in S\backslash\{i\}$ as the second most preferred school and are assigned to it with positive probability. Similarly, a positive measure of those who prefer j the most but not assigned to j have i as the second most preferred school and are assigned to it with positive probability. Hence, a positive measure of seats at every school are assigned to those who do not prefer that school the most.

Part (ii): Fix any $i \in S\backslash\{w\}$. Any student who prefers school w to school i, ranks w higher than school i. Since w is the worst school with cutoff score n, such a student is never assigned to school i. Consequently, any students who are assigned to school i are those who prefer i to w. Hence, almost all of those assigned to i under DA-MTB will be strictly worse off from getting reassigned w. This proves that there is no within-$\{i, w\}$ ex post reallocation of the DA-MTB allocation that Pareto dominates it. Hence, by Lemma **??**-(iii), the DA-MTB allocation is EPPO and EAPO within $\{i, w\}$ for any $i \in S\backslash\{w\}$.

Part (iii): Choose any two schools $\{i, j\}$ for $i, j \in S\backslash\{w\}$. Generically, there are a positive measure of students whose best school is i and the second best is j and a positive measure of students whose best school is j and the second best is i, respectively. Since neither school i nor j is a worst school, their cutoffs are strictly less than n. Hence, a positive measure of the former type is assigned to school j and a positive measure of the latter type is assigned to school i. Clearly, the ex post allocation from DA-MTB is not EPPO within $\{i, j\}$. The result follows from Lemma **??**-(i) and (iii).

Part (iv): This part follows directly from Part (iii) since any three schools include two schools that are not the worst school and since generically there is only one worst school. ▌

Proof of Theorem ??: Proof of Part (i) builds on that of part (ii), so it will appear last. Throughout, we let K and J be the sets of over- and under-subscribed schools.

Part (ii): Let $\nu^*(\cdot)$ be an equilibrium, and let $\phi^*(\cdot)$ be the ex ante allocation induced by $\nu^*(\cdot)$.

For any $\mathbf{v} \in \mathcal{V}$, consider an optimization problem:

$$[P(\mathbf{v})] \qquad \max_{\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}} \sum_{i \in S} v_i x_i$$

subject to

$$\sum_{i \in K} p_i x_i \leq \sum_{i \in K} p_i \phi_i^*(\mathbf{v}),$$

where $p_i \equiv \max\{\int \overline{\nu}_i^*(\tilde{\mathbf{v}}) d\mu(\tilde{\mathbf{v}}), 1\}$.

We first prove that $\phi^*(\mathbf{v})$ solves $[P(\mathbf{v})]$. This is trivially true for any $\mathbf{v}$ with $\pi_1(\mathbf{v}) = w$ since $x_i = \phi_i^*(\mathbf{v}) = 0, i \in K$ must hold by Lemma 3 (ii). Based on this observation, in the rest of the proof, we will restrict attention to students whose most preferred school is not the worst school.

Consider now any $\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}$ satisfying the constraint of $[P(\mathbf{v})]$, and suppose a type $\mathbf{v}$-student faces all others playing their parts of the equilibrium strategies ν^* under the original CADA game. Consider a strategy called s_i in which she picks school $i \in S$ as her target in her auxiliary message and submits it as her top choice in her ordinal list, and but submits truthful ordinal list otherwise. If type $\mathbf{v}$ plays strategy s_i, then she will be assigned to school i with probability

$$\frac{1}{\max\{\int \overline{\nu}_i^*(\tilde{\mathbf{v}}) d\mu(\tilde{\mathbf{v}}), 1\}} = \frac{1}{p_i}.$$

If $i \in J$, this probability is one. If $i \in K$, then she will be rejected by school i with positive probability. In that event, she will pass through the DA process according to her true ordinal preferences, and will be assigned based on her non-target draw of score ω_2. Since she will never be assigned to any other schools in K, she will only be assigned to a school in J. Which school in J she is assigned to is determined solely by ω_2 (holding fixed the student's ordinal rankings), and her draw of ω_2 is independent of her draw of ω_1 (which determined her assignment to i). Hence, the conditional probability of a student getting assigned to $j \in J$, is the same, regardless of which oversubscribed school $i \in K$ turned him down. Note let that conditional probability be $\bar{\phi}_j^*(\mathbf{v})$. Obviously, $\sum_{j \in J} \bar{\phi}_j^*(\mathbf{v}) = 1$.

In summary, when playing $s_i, i \in K$ she will be assigned to school $j \in J$ with probability

$$\left(1 - \frac{1}{p_i}\right) \bar{\phi}_j^*(\mathbf{v}).$$

Suppose now the type $\mathbf{v}$ student randomizes by choosing "strategy s_i" with probability $y_i := p_i x_i$, for each $i \in K$, and with probability

$$y_j := \nu_j^*(\mathbf{v}) + \left[\sum_{i \in K} (v_i^*(\mathbf{v}) - p_i x_i) \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}),$$

for each $j \in J$. Observe $y_j \geq 0$ for all $j \in S$. This is obvious for $j \in K$. For $j \in J$, this follows since the terms in the square brackets are nonnegative:

$$\begin{aligned}\sum_{i\in K}(v_i^*(\mathbf{v})-p_ix_i)\left(1-\frac{1}{p_i}\right) &= \sum_{i\in K}(p_i\phi_i^*(\mathbf{v})-p_ix_i)\left(1-\frac{1}{p_i}\right)\\ &= \left[\sum_{i\in K}p_i(\phi_i^*(\mathbf{v})-x_i)\right]-\left[\sum_{i\in K}(\phi_i^*(\mathbf{v})-x_i)\right]\\ &= \sum_{i\in K}p_i(\phi_i^*(\mathbf{v})-x_i)\\ &\geq 0,\end{aligned}$$

where the first inequality is implied by Lemma 3-(iii), the third equality holds since $\mathbf{x}\in\Delta^K_{\phi^*(\mathbf{v})}$, and the last inequality follows from the fact that $\mathbf{x}$ satisfies the constraint of $[P(\mathbf{v})]$. Further,

$$\begin{aligned}\sum_{i\in S}y_i &= \sum_{i\in K}p_ix_i+\sum_{j\in J}\left[\nu_j^*(\mathbf{v})+\left[\sum_{i\in K}(v_i^*(\mathbf{v})-p_ix_i)\left(1-\frac{1}{p_i}\right)\right]\bar{\phi}_j^*(\mathbf{v})\right]\\ &= \sum_{i\in K}p_ix_i+\sum_{j\in J}\nu_j^*(\mathbf{v})+\sum_{i\in K}\left[(\nu_i^*(\mathbf{v})-p_ix_i)\left(1-\frac{1}{p_i}\right)\right]\left(\sum_{i\in J}\bar{\phi}_j^*(\mathbf{v})\right)\\ &= \sum_{i\in K}p_ix_i+\sum_{j\in J}\nu_j^*(\mathbf{v})+\sum_{i\in K}\left[(\nu_i^*(\mathbf{v})-p_ix_i)\left(1-\frac{1}{p_i}\right)\right]\\ &= \sum_{i\in K}\nu_i^*(\mathbf{v})+\sum_{j\in J}\nu_j^*(\mathbf{v})+\sum_{i\in K}(\phi_i^*(\mathbf{v})-x_i)\\ &= \sum_{i\in S}\nu_i^*(\mathbf{v})=1.\end{aligned}$$

The third equality holds since $\sum_{i\in J}\bar{\phi}_i^*(\mathbf{v})=1$, the fourth is implied by Lemma **??**-(iii), and the fifth follows since $\mathbf{x}\in\Delta^K_{\phi^*(\mathbf{v})}$, (which implies $\sum_{i\in K}x_i=\sum_{i\in K}\phi^*(\mathbf{v})$).

By playing the mixed strategy $(y_1,...,y_n)$, the student is assigned to school $i\in K$ with probability

$$\frac{y_i}{p_i}=x_i,$$

and to each school $j \in J$ with probability

$$
\begin{aligned}
& y_j + \left[\sum_{i\in K} y_i \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}) \\
& = \nu_j^*(\mathbf{v}) + \left[\sum_{i\in K} (\nu_i^*(\mathbf{v}) - p_i x_i) \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}) + \left[\sum_{i\in K} p_i x_i \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}) \\
& = \nu_j^*(\mathbf{v}) + \left[\sum_{i\in K} \nu_i^*(\mathbf{v}) \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}) \\
& = \overline{\nu}_j^*(\mathbf{v}) + \left[\sum_{i\in K} \overline{\nu}_i^*(\mathbf{v}) \left(1 - \frac{1}{p_i}\right)\right] \bar{\phi}_j^*(\mathbf{v}) \\
& = \phi_j^*(\mathbf{v}) = x_j.
\end{aligned}
$$

In other words, the student $\mathbf{v}$ can replicate any $\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}$ that satisfies $\sum_{i\in K} p_i x_i \leq \sum_{i\in K} p_i \phi_i^*(\mathbf{v})$ by playing a strategy available in the CADA game. Since $\phi^*(\cdot)$ solves the CADA game and is still feasible in more constrained problem $[P(\mathbf{v})]$, it must solve $[P(\mathbf{v})]$. Moreover, since μ is atomless and $[P(\mathbf{v})]$ has a linear objective function on a convex set, $\phi^*(\mathbf{v})$ must be the unique solution to $[P(\mathbf{v})]$ for μ-a.e. $\mathbf{v}$.

We prove the statement of the theorem by contradiction. Suppose to the contrary that there exists an allocation $\phi(\cdot) \in \mathcal{X}^K_{\phi^*}$ that Pareto dominates $\phi^*(\cdot)$. Then, for μ-a.e. $\mathbf{v}$, $\phi(\mathbf{v})$ must either solve $[P(\mathbf{v})]$ or violate the constraint. For μ-a.e. $\mathbf{v}$, the solution to $[P(\mathbf{v})]$ is unique and coincides with $\phi^*(\mathbf{v})$. Therefore, we must have

$$
\sum_{i\in K} p_i \phi_i(\mathbf{v}) \geq \sum_{i\in K} p_i \phi_i^*(\mathbf{v}), \tag{6}
$$

for μ-a.e. $\mathbf{v}$. Further, there must exist a set $A \subset \mathcal{V}$ with $\mu(A) > 0$ such that each student $\mathbf{v} \in A$ must strictly prefer $\phi(\mathbf{v})$ to $\phi^*(\mathbf{v})$, which must imply (since $\phi^*(\mathbf{v})$ solves $[P(\mathbf{v})]$)

$$
\sum_{i\in K} p_i \phi_i(\mathbf{v}) > \sum_{i\in K} p_i \phi_i^*(\mathbf{v}), \ \forall \mathbf{v} \in A. \tag{7}
$$

Combining (**??**) and (**??**), we get

$$
\begin{aligned}
& \int \sum_{i\in K} p_i \phi_i(\mathbf{v}) d\mu(\mathbf{v}) > \int \sum_{i\in K} p_i \phi_i^*(\mathbf{v}) d\mu(\mathbf{v}) \\
\Leftrightarrow & \sum_{i\in K} p_i \int \phi_i(\mathbf{v}) d\mu(\mathbf{v}) > \sum_{i\in K} p_i \int \phi_i^*(\mathbf{v}) d\mu(\mathbf{v}).
\end{aligned} \tag{8}
$$

Now since $\phi(\cdot) \in \mathcal{X}$, for each $i \in S$,

$$
\int \phi_i(\mathbf{v}) d\mu(\mathbf{v}) = 1 = \int \phi_i^*(\mathbf{v}) d\mu(\mathbf{v}).
$$

Multiplying both sides by p_i and summing over K, we get

$$\sum_{i \in K} p_i \int \phi_i(\mathbf{v}) d\mu(\mathbf{v}) = \sum_{i \in K} p_i \int \phi_i^*(\mathbf{v}) d\mu(\mathbf{v}),$$

which contradicts (**??**). We thus conclude that ϕ^* is Pareto optimal within K.

Part (iii): Consider the following maximization problem for every $\mathbf{v} \in \mathcal{V}$:

$$[\overline{P}(\mathbf{v})] \qquad \max_{\mathbf{x} \in \Delta} \sum_{i \in S} v_i x_i$$

subject to

$$\sum_{i \in K} p_i x_i \leq 1. \tag{9}$$

When we have only one undersubscribed school, called school n, the allocation x_n is completely pinned down by the allocation among $n-1$ oversubscribed schools, that is,

$$x_n = 1 - \sum_{i \in K} x_i.$$

Therefore, an allocation $\mathbf{x} \in \Delta$ is feasible in CADA game if (and only if) (**??**) holds.

Now consider the following maximization problem:

$$[\overline{P}'(\mathbf{v})] \qquad \max_{\mathbf{x} \in \Delta} \sum_{i \in S} v_i x_i$$

subject to

$$\sum_{i \in K} p_i x_i \leq \sum_{i \in K} p_i \phi_i^*(\mathbf{v}). \tag{10}$$

Since $\phi^*(\cdot)$ solves less constrained problem $[\overline{P}(\mathbf{v})]$ and is still feasible in $[\overline{P}'(\mathbf{v})]$, it must be an optimal solution for $[\overline{P}'(\mathbf{v})]$. The rest of the proof is shown by the same argument as in Part (i).

Part (i): Let ψ^* be the ex post allocation resulting from a CADA equilibrium. Suppose to the contrary that there is an allocation $\tilde{\psi} \in \mathcal{Y}$ that Pareto dominates ψ^*. Consider first students with $\theta = (\mathbf{v}, \omega)$ assigned to an (oversubscribed) school $i \in K$. They must have named school i for their target (or else they would never have been assigned to it). Almost all such types of students strictly prefer i to every $j \in J$ (by Lemma **??**-(ii)). For $\tilde{\psi}$ to Pareto dominate ψ^*, we must have $\tilde{\psi}_h(\theta) = 1$ for some $h \in K$, for almost all such θ. In other words, almost all students assigned to an oversubscribed school under ψ^* must be assigned an oversubscribed school under $\tilde{\psi}$. Then,

$$|K| \geq \sum_{i \in K} \eta(\{\theta | \tilde{\psi}_i(\theta) = 1\}) \geq \sum_{i \in K} \eta(\{\theta | \psi_i^*(\theta) = 1\}) = |K|.$$

It follows that almost all students assigned to an undersubscribed school under ψ^* must also be assigned to an undersubscribed school under $\tilde{\psi}$.

If $\tilde{\psi}$ (with this property) Pareto dominates ψ^*, there must exist a within-K ex post reallocation or an within J ex post reallocation of ψ^* that Pareto dominates ψ^*. Part (ii), along with Lemma **??**-(ii), implies that ψ^* is EPPO within K. Hence, there is no within-K ex post reallocation of ψ^* that Pareto dominates ψ^*. It now suffices to show that there is no within-J reallocation of ψ^* that Pareto dominates it. This part of the proof parallels that of Theorem **??**-(ii), so it is omitted. ▌

Proof of Lemma ??. We can show that, if a school $j > 1$ is oversubscribed, then school $j-1$ is oversubscribed. (Those who pick j as target should have picked j, giving a contradiction.)

It then suffices to show that at least schools $\{1, 2\}$ are oversubscribed. Suppose not. Then, only school 1 is oversubscribed in equilibrium. Suppose mass $m_2 < 1$ of students pick school 2 as their target; and all other mass $n - m_2$ pick school 1 as their target. (No student picks school $j > 2$, since picking school 2 will guarantee enrollment, which dominates choosing any school $j > 2$.) Pick any student with $\mathbf{v}$ such that $\frac{\sum_{i=1}^{n} v_i}{n} < v_2$. If the student picks school 2, she can guarantee the payoff of v_2. If the student picks school 1, she can get

$$v_1 \frac{1}{n-m_2} + v_2 \frac{1-m_2}{n-m_2} + (\sum_{i=3}^{n} v_i) \frac{1}{n-m_2} = (\sum_{i=1}^{n} v_i) \frac{1}{n-m_2} - v_2 \frac{m_2}{n-m_2},$$

which is less than v_2. Hence, all such students must be choosing 2 as their target. Since there is more than unit mass of such students, school 2 cannot be undersubscribed, which contradicts the hypothesis that only school 1 is oversubscribed. ▌

Proof of Theorem ??: Consider first a DA algorithm with any random tie-breaking. Since all students submit the same rank order over schools, they all must be assigned to each school with the same probability. In other words, the allocation must be

$$\phi^{DA}(\mathbf{v}) = (\frac{1}{n}, ..., \frac{1}{n}) \text{ for all } \mathbf{v}.$$

Consider now CADA algorithm. Let $\nu^*(\mathbf{v}) \in \Delta$ be the equilibrium mixed strategy adopted by type $\mathbf{v}$. Then, a measure

$$\alpha_i^* := \int \nu_i^*(\mathbf{v}) d\mu(\mathbf{v})$$

of students pick $i \in S$ for their target in equilibrium. The equilibrium induces a mapping $\varphi^* : S \mapsto \Delta$, whereby a student is assigned to school j with probability $\varphi_j^*(i)$ if she picks i for her target.

Since in equilibrium, the capacity of each school is filled, we must have, for each $j \in S$,

$$\sum_{i \in S} \alpha_i^* \varphi_j^*(i) = 1. \tag{11}$$

That is, a measure α_i^* of students pick i for their target, and a fraction $\varphi_j^*(i)$ of those is assigned to school j. Summing the product over all i then gives the measure of students assigned to j, which must equal its capacity, 1.

Consider a student with any arbitrary $\mathbf{v} \in \mathcal{V}$. Suppose she randomizes in her auxiliary message by choosing school i with probability

$$y_i := \frac{\alpha_i^*}{\sum_j \alpha_j^*} = \frac{\alpha_i^*}{n}.$$

Then, the probability that she will be assigned to any school k is

$$\sum_j y_j \varphi_k^*(j) = \sum_j \frac{\alpha_j^*}{n} \varphi_k^*(j) = \frac{1}{n},$$

where the second equality follows from (**??**). That is, she can replicate the same ex ante assignment with the randomization strategy as $\phi^{DA}(\mathbf{v})$. Hence, the student must be at least weakly better from CADA. This proves the first statement.

We next prove the second statement. There are two cases. Suppose first school 2 is undersubscribed. Then, each student with $\mathbf{v} \in \mathcal{V}_2^U$ can pick school 2 for her target and guarantee assignment to school 2. The resulting payoff, v_2, exceeds $\sum_{i \in S} \frac{v_i}{n}$ (since $\mathbf{v} \in \mathcal{V}_2^U$), so every such student must be strictly better off in CADA than in the DA and we are done. Suppose therefore school 2 is oversubscribed. Then, school 1 must be also oversubscribed. This means that $\alpha_2^* \in [1, n)$ in the CADA equilibrium.

[TO BE COMPLETED.] ▌

Proof of Theorem ??: Part (i) is precisely the same as Part (i) of Theorem **??** and is the consequence of Part (ii) below and Part (ii) of Lemma **??** (which does not depend on whether the students are naive or not). Hence it is omitted. To prove Part (ii), it is useful to establish the following lemma. As before, let ϕ^* denote the ex ante allocation arising from the CADA game and let K and $J = S \backslash K$ be respectively the sets of oversubscribed and undersubscribed schools in equilibrium.

Lemma N: *Any reassignment of $\phi^*(\mathbf{v})$ within K will make a naive student with $\mathbf{v}$ strictly worse off, for almost every $\mathbf{v}$.*

Proof: Consider a naive student with $\mathbf{v}$. Assume without loss of generality that she prefers i strictly over all other schools (i.e., $v_i > v_j$, $\forall j \neq i$). (This is without loss of generality since the values are distinct for almost every student type.) Since the student is naive, she subscribes to school i with probability 1. If school i is undersubscribed, then the result is trivial since $\phi_k^*(\mathbf{v}) = 0$ for all $k \in K$. Hence, suppose school i is oversubscribed. Then, any reassignment $\mathbf{x} \in \Delta_{\phi^*(\mathbf{v})}^K$ must satisfy

$$\sum_{j \in K} x_j = \phi_i^*(\mathbf{v})$$

Since $v_i > v_j$ $\forall j \neq i$, for any $\mathbf{x} \in \Delta_{\phi^*(\mathbf{v})}^K$, $\mathbf{x} \neq \phi_i^*(\mathbf{v})$, we must have

$$\sum_{j \in K} x_j v_j < \sum_{j \in K} x_j v_i = \phi_i^*(\mathbf{v}) v_i,$$

which implies that the student must be strictly worse off from any such reassignment. $\|$

We are now ready to prove Parts (ii) and (iii):

Part (ii): We make use of the proof of Theorem **??**. By Lemma N, a type-$\mathbf{v}$ naive student's assignment from the CADA, $\phi^*(\mathbf{v})$, is a unique solution to $[P(\mathbf{v})]$, for a.e. $\mathbf{v}$, even without the constraint

$$\sum_{i \in K} p_i x_i \leq \sum_{i \in K} p_i \phi_i^*(\mathbf{v}). \tag{12}$$

Since $\phi^*(\mathbf{v})$ is feasible under (**??**), this must be a unique solution to $[P(\mathbf{v})]$.

For a non-naive student with a.e. $\mathbf{v}$, the proof of Theorem **??** follows directly, so $\phi^*(\mathbf{v})$, is also a unique solution of $[P(\mathbf{v})]$. Since the equilibrium assignment of both types solves $[P(\mathbf{v})]$, the rest of the argument in the proof of Theorem **??** applies, proving that we ϕ^* is EAPO within K. $\|$

Part (iii): Again let ϕ^* be the ex ante allocation arising from CADA. Suppose to the contrary that there exists a within-$K \cup \{l\}$ reallocation $\tilde{\phi}$ of ϕ^* that Pareto dominates ϕ^*. By Part (ii), ϕ^* is EAPO within K, so $\tilde{\phi}_l(\mathbf{v}) \neq \phi_l^*(\mathbf{v})$ for a positive measure of $\mathbf{v}$, which in turn implies that there exists a set $A \subset \mathcal{V}$ with $\mu(A) > 0$ such that $\tilde{\phi}_l(\mathbf{v}) > \phi_l^*(\mathbf{v})$ for each $\mathbf{v} \in A$. Since $\tilde{\phi}(\mathbf{v}) \in \Delta_{\phi^*(\mathbf{v})}^{K \cup \{l\}}$, $\sum_{j \in K \cup \{l\}} \tilde{\phi}_j(\mathbf{v}) = \sum_{j \in K \cup \{l\}} \phi_j^*(\mathbf{v})$, so

$$\sum_{j \in K} \tilde{\phi}_j(\mathbf{v}) < \sum_{j \in K} \phi_j^*(\mathbf{v}) \quad \text{for all } \mathbf{v} \in A.$$

Assume without loss that $\mathbf{v}$ satisfies $v_i > v_j$ for all $i \in K$ and for all $j \neq i$. ($i \neq l$ since $\tilde{\phi}_l(\mathbf{v}) > \phi_l^*(\mathbf{v})$ is impossible if $i = l$.) Then, the type-$\mathbf{v}$ student's expected payoff from $\tilde{\phi}$ is

$$\begin{aligned}
\sum_{j \in S} \tilde{\phi}_j(\mathbf{v}) v_j &= \sum_{j \in K} \tilde{\phi}_j(\mathbf{v}) v_j + \tilde{\phi}_l(\mathbf{v}) v_l + \sum_{j \in J \setminus \{l\}} \phi_j^*(\mathbf{v}) v_j \\
&< \sum_{j \in K} \tilde{\phi}_j(\mathbf{v}) v_i + \left(\tilde{\phi}_l(\mathbf{v}) - \phi_l^*(\mathbf{v})\right) v_i + \phi_l^*(\mathbf{v}) v_l + \sum_{j \in J \setminus \{l\}} \phi_j^*(\mathbf{v}) v_j \\
&= \left(\sum_{j \in K \cup \{l\}} \tilde{\phi}_j(\mathbf{v}) - \phi_l^*(\mathbf{v}) \right) v_i + \sum_{j \in J} \phi_j^*(\mathbf{v}) v_j \\
&= \sum_{j \in S} \phi_j^*(\mathbf{v}) v_j.
\end{aligned}$$

Since this inequality holds for almost every $\mathbf{v} \in A$, and since $\mu(A) > 0$, $\tilde{\phi}$ cannot Pareto dominate ϕ^*. ▌

Appendix C: Simulations

There are 5 schools each with a capacity of 20 seats and 100 students. Fix α. We independently draw 100 sets of vNM values for students. Let $\{v_{ij}^s\}$ denote a draw of vNM values, where superscript s denote the draw and v_{ij}^s denotes student i's vNM value for school j.

Given a draw $\{v_{ij}^s\}$, fix the mechanism, define the following: p_{ij}^s is the probability that student i is assigned school j under the mechanism. $r_i^s(k)$ is the school that is ranked k-th in i's preference list. P^s is the set of popular schools. O^s is the set of oversubscribed schools in an equilibrium of CADA with no naive players. We normalize/mean-adjust the vNM values in our welfare calculations. For that, we use $\{\hat{v}_{ij}\}$ in our computations, which is defined by

$$\hat{v}_{ij} = v_{ij} - \frac{1}{500}\sum_{i'}\sum_{j'} v_{i'j'}$$

By abusing our notation, we will refer the *mean-adjusted* vNM values by $\{v_{ij}^s\}$ hereafter.

Given the mechanism, for each draw s, we calculate the following:

$$\overline{v}^s = \frac{1}{100}\sum_i\sum_j p_{ij}^s v_{ij}^s$$
$$\overline{v}_H^s = \frac{1}{100}\frac{5}{|H|}\sum_i\sum_{j\in H} p_{ij}^s v_{ij}^s \text{ for } H \in \{P^s, M^s, O^s\}$$

where $\overline{v}^s$ is the mean utility of all students and $\overline{v}_H^s$ is the mean utility of students assigned to a school in $H \in \{P^s, M^s, O^s\}$, and $|H|$ is the cardinality of H.

A first best or utilitarian maximum solves

$$\overline{v}_{FB}^s = \max_{\{\hat{p}_{ij}^s\}}\sum_i\sum_j \hat{p}_{ij}^s v_{ij}^s$$

Let $\{\bar{p}_{ij}^s\}$ denote a solution to the first best. There may be multiple solutions, we arbitrarily pick one. We also compute mean-utility of students a certain type of schools at the first best as follows:

$$\overline{v}_{FB|H}^s = \frac{1}{100}\frac{5}{|H|}\sum_i\sum_{j\in H} \bar{p}_{ij}^s v_{ij}^s \text{ for } H \in \{P^s, M^s, O^s\}$$

Furthermore, we calculate

$$\pi_1^s = \frac{1}{100}\sum_i\sum_j p_{ij}^s.\mathbf{1}(r_i^s(1) = j)$$
$$\pi_{12}^s = \frac{1}{100}\sum_i\sum_j p_{ij}^s.\mathbf{1}(r_i^s(1) = j \text{ or } r_i^s(2) = j)$$

where π_1^s is the average probability of assigning a student her first choice and π_{12}^s is the average probability of assigning a student her first or second choice. Note that we calculate these numbers for the first best as well.

In the CADA experiments with naive players, we divide the set of students into two: N is the set of naive players who always pick their first choice as target, and S is the set of strategic/sophisticated players who play their best response strategies given others' strategies. We calculate the mean-utilities accordingly.

We also compute the number of students picking their k-th choice as their target school in equilibrium, which we denote by T_k^s, $k \in \{1, 2, 3, 4\}$

Given a draw $\{v_{ij}^s\}$, the sets P^s and M^s are determined trivially. Next we describe how the other numbers are computed.

A *single tie breaker* is a list of 100 randomly drawn numbers, one for each student. Under DA-STB the ties at a school are broken according to students' single random numbers. In CADA, we draw two single tie breakers, one to be used to break ties at one's target school, the other to be used at one's other schools. A *multiple tie breaker* is a list of $100 \times 5 = 500$ randomly drawn numbers, one for each student at each school. Under DA-MTB, the ties at a school are broken according to students' tie breaker numbers at that school.

For each draw $\{v_{ij}^s\}$, we independently draw 2,000 single tie breakers for DA-STB, and an additional set of 2,000 single tie breakers for CADA, and 2,000 multiple tie breakers for DA-MTB. Then p_{ij}^s for a mechanism is computed by

$$\frac{\text{Number of tie breakers at which } i \text{ is assigned } j}{2,000}$$

The equilibrium of CADA is computed with single tie breakers being fixed. Given the strategies of other students, a student's best response is found by computing that student's expected utility over those tie breakers. Then O^s, the set of oversubscribed schools, is found by using students's equilibrium target schools. In experiments with naive players, naive players' target schools are fixed at their first choice.

Note that we are approximating the equilibrium by drawing (two sets of) 2,000 independent tie-breakers. The exact numbers are computed by considering 100! single tie-breakers and $(100!)^5$ multiple tie breakers, which is beyond the capabilities of our computational resources. Also, we are picking an equilibrium arbitrarily from a set of possibly multiple equilibria.

For each $z^s \in \{\overline{v}^s, \overline{v}_{O^s}^s, \overline{v}_{FB}^s, \overline{v}_{FB|O^s}^s, \pi_1^s, \pi_{12}^s, T_1^s, T_2^s, T_3^s, T_4^s, |P^s|, |O^s|\}$, we compute the average of z^s by

$$z = \frac{1}{100} \sum_{s=1}^{100} z^s$$

Note that we drop all "s" from a variable to denote its mean over 100 iterations of an experiment.

We report the following: $100\frac{\overline{v}}{\overline{v}_{FB}}$ is the percentage welfare with respect to to the utilitarian maximum; $100\frac{\overline{v}_O}{\overline{v}_{FB|O}}$ is the percentage welfare with respect to the utilitarian maximum at oversubscribed schools. $100\pi_1$ is the percentage of students getting their first choice. $|P|$ and $|O|$ are the number of popular and oversubscribed schools, respectively. $|T_k|$ is the number of students picking their k-th choice as target school in equilibrium.